

Problems of unique determination of domains by the relative metrics on their boundaries *

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December 13, 2016

Abstract

This survey is devoted to discussing the problems of the unique determination of surfaces that are the boundaries of (generally speaking) nonconvex domains.

First (in Sec. 2) we examine some results on the problem of the unique determination of domains by the relative metrics of the boundaries. Then, in Sec. 3, we study rigidity conditions for the boundaries of submanifolds in a Riemannian manifold. The final part (Sec. 4) is concerned with the unique determination of domains by the condition of the local isometry of boundaries in the relative metrics.

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*Mathematical Subject Classification (2010). 53C45 (primary); Key words: intrinsic metric, relative metric of boundary, local isometry of boundaries, strict convexity

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1 Introduction

The topic of the article, although relatively new, has a straightforward connection to the classical problems of a bicentennial history. As the starting point we may view the celebrated Cauchy theorem which claims that a convex polyhedron is uniquely determined from its unfolding. Later, the problems of unique determination of convex surfaces were studied by Minkowski, Hilbert, Weyl, Blaschke, Cohn-Vossen, and other prominent mathematicians. The greatest progress in this direction was achieved by A. D. Alexandrov and his students. Mention Pogorelov's celebrated theorem on unique determination of a closed convex surface from its intrinsic metric (for example, see [1]).

In [2], we proposed a new approach to the problem of unique determination of surfaces, which allowed to substantially enlarge the framework of the problem. The following model situation illustrates the essence of this approach fairly well.

Let U_1 and U_2 be two domains (i.e., open connected sets) in the real n -dimensional Euclidean space \mathbb{R}^n whose closures $\text{cl } U_j$, where $j = 1, 2$, are Lipschitz manifolds (such that $\partial(\text{cl } U_j) = \partial U_j \neq \emptyset$ where ∂E is the boundary of E in \mathbb{R}^n). Assume also that the boundaries ∂U_1 and ∂U_2 of these domains, which coincide with the boundaries of the manifolds $\text{cl } U_1$ and $\text{cl } U_2$, are isometric with respect to their relative metrics $\rho_{\partial U_j, U_j}$ ($j = 1, 2$), i.e., with respect to the metrics that are the restrictions to the boundaries ∂U_j of the extensions (by continuity) of the intrinsic metrics of the domains U_j to $\text{cl } U_j$. The following natural question arises: *Under which additional conditions are the domains U_1 and U_2 themselves isometric (in the Euclidean metric)?* In particular, the natural character of this problem is determined by the circumstance that the problem of unique determination of closed convex surfaces mentioned in the beginning of the article is its most important particular case. Indeed, assume that S_1 and S_2 are two closed convex surfaces in \mathbb{R}^3 , i.e., they are the boundaries of two bounded convex domains $G_1 \subset \mathbb{R}^3$ and $G_2 \subset \mathbb{R}^3$. Let $U_j = \mathbb{R}^3 \setminus \text{cl } G_j$ be the complement of the closure $\text{cl } G_j$ of the domain G_j , $j = 1, 2$. Then the intrinsic metrics on the surfaces $S_1 = \partial U_1$ and $S_2 = \partial U_2$ coincide with the relative metrics $\rho_{\partial U_1, U_1}$ and $\rho_{\partial U_2, U_2}$ on the boundaries of the domains U_1 and U_2 , and thus, the problem of unique determination of closed convex surfaces by their intrinsic metrics is indeed a particular case of the problem of unique determination

of domains by the relative metrics on their boundaries.

The generalization of the problem of the unique determination of surfaces ensuing from a new approach suggested in [2] manifests itself in the fact that the unique determination of domains by the relative metrics on their boundaries holds not only when their complements are bounded convex sets but, for example, also in the following cases.

The domain U_1 is bounded and convex, while the domain U_2 is arbitrary (A. P. Kopylov (see [2])).

The domain U_1 is strictly convex and the domain U_2 is arbitrary (A. D. Aleksandrov (see [3])).

The domains U_1, U_2 are bounded and their boundaries are smooth (V. A. Aleksandrov (see [3])).

The domains U_1, U_2 have nonempty bounded complements, while their boundaries are $(n - 1)$ -dimensional connected C^1 -manifolds without boundary, $n > 2$ (V. A. Aleksandrov (see [4])).

Recall some definitions of unique determination that are needed below (for example, see [5]).

Definition 1.1. Let \mathcal{U} be a class of domains (i.e., open connected sets) in real Euclidean n -dimensional space \mathbb{R}^n , where $n \geq 2$. We say (see, e.g., [5]) that a domain $U \in \mathcal{U}$ is uniquely determined in the class \mathcal{U} by the relative metric of its (Hausdorff) boundary if each domain $V \in \mathcal{U}$ whose Hausdorff boundary is isometric to the Hausdorff boundary of the domain U with respect to the relative metrics is itself isometric to U (with respect to Euclidean metric).

Remark 1.1. Suppose that U is a domain in \mathbb{R}^n ($n \geq 2$) and ρ_U is its intrinsic metric.¹ Consider the Hausdorff completion (U_H, ρ_{U_H}) of the metric space (U, ρ_U) , i.e., the completion of this space in intrinsic metric ρ_U . Identifying the points of this completion that correspond to points of the domain U with these points themselves and removing them from the completion, we obtain a metric space $(\partial_H U, \rho_{\partial_H U, U})$; the set $\partial_H U$ of its elements is called the Hausdorff boundary of the domain U , and $\rho_{\partial_H U, U}$ is the relative metric on this Hausdorff boundary. The isometry of the Hausdorff boundaries of domains U and V with respect to their relative metrics means the existence of a surjective isometry $f : (\partial_H U, \rho_{\partial_H U, U}) \rightarrow (\partial_H V, \rho_{\partial_H V, V})$ between these boundaries.

Remark 1.2. If the intrinsic metric ρ_U of a domain U extends by continuity to the closure $\text{cl}U$ then $\partial_H U$ is naturally identified with the

¹Recall that the distance between points $x, y \in U$ in the intrinsic metric of U equals the infimum of the lengths of the curves joining x and y within U .

Euclidean boundary ∂U .

It is also worth noting that M. V. Korobkov obtained a complete description of domains that are uniquely determined in the class of all domains by the condition of the (global) isometry of their boundaries in the relative metrics (see [6], [7], [8], [9]²). In [6], [7], he obtained the following results.

Henceforth $\text{Int } E$ is the interior of a set E , while $\text{cl } E$ is the closure of E , ∂E is the boundary of E , and $\text{conv } E$ is the convex hull of E .

Given a domain U , we put $F_U = U \setminus \text{cl}(\text{conv } \partial U)$ and denote by U_i the connected components of the open set $U \cap \text{Int}(\text{conv } \partial U)$.

Theorem 1.1 *Let U be a domain in \mathbb{R}^2 . Then:*

(I) *If ∂U lies on some straight line then U cannot be uniquely determined (in the class of all domains in \mathbb{R}^2) if and only if ∂U is a connected set containing more than one point;*

(II) *If there is no straight line containing ∂U then U cannot be uniquely determined if and only if there exist a domain V nonisometric to U , a family of isometric mappings $Q_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, and a homeomorphism $\theta : \partial F_U \rightarrow \partial F_V$ satisfying the following conditions:*

(IIa) $Q_i(U_i) = V_i$ for each component U_i . The same equality is also valid for each component V_i .

(IIb) $x \in U \cap \partial U_i$ if and only if $Q_i(x) \in V \cap \partial V_i$; the validity of these containments implies that $\theta(x) = Q_i(x)$.

(IIc) *The homeomorphism θ preserves the arc length (i.e., for arbitrary two points x and y the length of the arc joining x and y within ∂F_U coincides with the length of the image of this arc under the mapping θ).*

As a straightforward consequence of Theorem 1.1 we have

Corollary 1.1 *Suppose that a domain $U \subset \mathbb{R}^2$ satisfies one of the following two conditions:*

(1) $F_U = \emptyset$;

(2) *the set F_U is nonempty and disconnected.*

Then U is uniquely determined from the relative metric of its Hausdorff boundary.

In turn, Corollary 1.1 contains the following particular case.

²In these works, the isometry of the Hausdorff boundaries of domains U and V with respect to their relative metrics means the existence of a bijective isometry $f : (\partial_H U, \rho_{\partial_H U, U}) \rightarrow (\partial_H V, \rho_{\partial_H V, V})$ between these boundaries.

Corollary 1.2 *Suppose that a domain $U \subset \mathbb{R}^2$ is bounded. Then U is uniquely determined from the relative metric of its Hausdorff boundary.*

The following theorem generalized A. D. Aleksandrov's theorem about the unique determination of a strictly convex domain in the class of domains whose intrinsic metrics extend by continuity to their closure (see [3]) to the case of convex domains. In the plane case, this theorem ensues from Corollary 1.1.

Theorem 1.2 *Each convex domain $U \subset \mathbb{R}^n$, $n \geq 2$, different from an open half-space in \mathbb{R}^n is uniquely determined from the relative metric of its Hausdorff boundary.*

In [8], [9], Korobkov obtained a result which is analogous to Theorem 1.1 and contains a complete description domains in \mathbb{R}^n , $n \geq 3$, that are uniquely determined in the class of all n -dimensional domains by the condition of the (global) isometry of their Hausdorff boundaries in the relative metrics.

In Section 2, we consider the problems of unique determination of domains by the condition of the (global) isometry of their boundaries in the relative metrics. We state of the proofs of Theorems 1.1, 1.2.

In this connection, there appears the following question: Is it possible to construct an analog of the theory of rigidity of surfaces in Euclidean spaces in the general case of the boundaries of submanifolds in Riemannian manifolds?

Section 3 of our article is devoted to a detailed discussion of this question. In it, we in particular obtain new results concerning rigidity problems for the boundaries of n -dimensional connected submanifolds with boundary in n -dimensional smooth connected Riemannian manifolds without boundary ($n \geq 2$).

Results of [6], [7], [8], [9] imply, in particular, that any bounded domain in \mathbb{R}^n is uniquely determined by the condition of isometry of boundaries in the relative metrics. At the same time, according to results of [10], a bounded polygonal plane domain U is uniquely determined by the condition of local isometry of boundaries in the relative metrics in the class of all such domains if and only if the domain U is convex.

Remark 1.3. Let \mathcal{M} be a class of domains in space \mathbb{R}^n with $n \geq 2$. Following [5], we say that a domain $U \in \mathcal{M}$ is uniquely determined in the class \mathcal{M} by the condition of local isometry of the (Hausdorff) boundaries of domains in the relative metrics if, for any domain V belonging to the class \mathcal{M} , the local isometry of its Hausdorff boundary to the Hausdorff boundary

of the domain U with respect to the relative metrics implies the isometry of the domains U and V (with respect to the Euclidean metric). The local isometry in the relative metrics between the Hausdorff boundaries $\partial_H U$ and $\partial_H V$ of the domains U and V means the existence of a bijective mapping $f : \partial_H U \rightarrow \partial_H V$ of these boundaries which is a local isometry with respect to their relative metrics, i.e., a mapping such that, for any element $y \in \partial_H U$, there exists a number $\varepsilon > 0$ satisfying the following condition: for any two elements a and b from the ε -neighborhood $Z(y) = \{z \in \partial_H U : \rho_{\partial_H U, U}(z, y) < \varepsilon\}$ of y , $\rho_{\partial_H U, U}(a, b) = \rho_{\partial_H V, V}(f(a), f(b))$. It is clear that f^{-1} is also a local isometry with respect to relative metrics of boundaries.

In Section 4 of this paper, we continue the study of the unique determination of domains by the condition of local isometry of their boundaries in the relative metrics.

It can be divided into two parts.

The first of them is mainly devoted to finding a complete description of conditions that are necessary and sufficient for a plane domain with smooth boundary to be uniquely determined by the condition of local isometry of their boundaries in the class of all domains with smooth boundaries (in the case of a bounded domain, in the class of all bounded plane domains with smooth boundaries).

In the second part of Section 4, we obtain some new assertions on the unique determination of space domains with smooth boundaries by the considering in the Section condition. All of these results emphasize the specific character of our approach to the problems of rigidity of domains in \mathbb{R}^n .

Note that below $[a, b] = \{bt + (1 - t)a \in \mathbb{R}^n : 0 \leq t \leq 1\}$, $[a, b[= \{bt + (1 - t)a \in \mathbb{R}^n : 0 \leq t < 1\}$ ($]a, b] = \{bt + (1 - t)a \in \mathbb{R}^n : 0 < t \leq 1\}$) and $]a, b[= \{bt + (1 - t)a \in \mathbb{R}^n : 0 < t < 1\}$ are the segment (closed interval), the half-open interval and the (open) interval in \mathbb{R}^n with endpoints $a, b \in \mathbb{R}^n$, $a \neq b$. $\text{Int } I$ is the interior of the segment (of the half-open interval) I , $\text{Int }]a, b[=]a, b[$. $B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$ is the open ball in \mathbb{R}^n of radius r ($0 < r < \infty$) centered at $x_0 \in \mathbb{R}^n$. Id_E is the identity mapping of a set E : $\text{Id}_E(x) = x$ for all $x \in E$.

In what follows, all paths (curves) $\gamma : [\alpha, \beta] \rightarrow \mathbb{R}^n$, where $\alpha, \beta \in \mathbb{R}$, are assumed continuous and non-constant, and $l(\gamma)$ means the length of a path γ . If $\gamma : [\alpha, \beta] \rightarrow \mathbb{R}^n$ is continuous and injective then γ is also called an arc.

2 On Unique Determination of Domains by Relative Metric of Boundaries

Below by connectedness we mean connectedness in the sense of general topology.

The support of an element \tilde{x} of the Hausdorff boundary $\partial_H U$ of a domain $U \subset \mathbb{R}^n$, $n \geq 2$, is a point x of the Euclidean boundary ∂U which is the limit of a Cauchy sequence (in the intrinsic metric of U) of points $x \in U$ representing \tilde{x} . We denote the support of $\tilde{x} \in \partial_H U$ by $p\tilde{x}$ ($= p_H \tilde{x}$).

It is clear that each element $\tilde{x} \in \partial_H U$ possesses the unique support $x = p\tilde{x}$. At the same time, there can be points $x \in \partial U$ that are not the support of any element of $\partial_H U$; on the other hand, there can be points $x \in \partial U$ that are the supports of several (even uncountable many) elements of $\partial_H U$. The following simple facts are valid:

Lemma 2.1 *Let U be a domain in \mathbb{R}^n . Then $\{p\tilde{x} : \tilde{x} \in \partial_H U\}$ is everywhere dense in ∂U (with respect to the usual Euclidean metric).*

Lemma 2.2 *If $\tilde{x} \in \partial_H U$ then the support $x = p\tilde{x}$ is attainable from U along a rectifiable curve $\gamma : [0, 1] \rightarrow \text{cl } U$ such that $\gamma(1) = x$, $\gamma(t) \in U$ for $0 \leq t < 1$, and, for each sequence $\{t_j\}_{j \in \mathbb{N}}$ of points $t_j \in [0, 1[$ satisfying the condition $\lim_{j \rightarrow \infty} t_j = 1$, the sequence $\{\gamma(t_j)\}_{j \in \mathbb{N}}$ is Cauchy in the intrinsic metric of U and represents the boundary element \tilde{x} . On the other hand, if a point $x \in \partial U$ is attainable from U along a rectifiable curve γ then every sequence $\{\gamma(t_j)\}_{j \in \mathbb{N}}$, where $t_j \in [0, 1[$, $j = 1, 2, \dots$, and $\lim_{j \rightarrow \infty} t_j = 1$, is Cauchy in the intrinsic metric of U and determines the only element $\tilde{x} \in \partial_H U$ whose support is x .*

We need some additional notions. Granted an ordered triple x, y, z of points, denote by $\angle(x, y, z)$ the plane angle $\angle(x, y, z) = \{y + t(x - y) + s(z - y) : t \geq 0, s \geq 0\}$; and by $\Delta(x, y, z)$, the (closed) triangle with vertices x, y , and z . We denote by $|\angle(x, y, z)|$ the value of the angle $\angle(x, y, z)$ (in radians).

Let U be a domain in \mathbb{R}^n different from the whole \mathbb{R}^n (we further consider only the domains of this kind). An interval $]x, y[$ is called a boundary interval for U if $\emptyset \neq]x, y[\subset U$ and $x, y \in \partial U$. We say that an ordered triple x, y, z of points determines a boundary angle for the domain U if these points do not lie on one straight line, $x, y, z \in \partial U$, $]x, y[\subset U$, $]y, z[\subset U$, and $\exists r > 0$ $B(y, r) \cap \angle(x, y, z) \setminus \{y\} \subset U$. Henceforth we say that a triangle $\Delta(x, y, z)$ is a boundary triangle for U if x, y , and z do not lie on one straight line,

$x, y, z \in \partial U$, and $\Delta(x, y, z) \setminus \{x, y, z\} \subset U$. It is obvious that each side of a boundary triangle is a boundary interval and the angles of this triangle are boundary angles.

Also, it is easily seen that each boundary interval $]x, y[$ naturally generates a pair $\tilde{x}, \tilde{y} \in \partial_H U$ of elements of the Hausdorff boundary such that $p\tilde{x} = x$, $p\tilde{y} = y$, and $\rho_{\partial_H U, U}(\tilde{x}, \tilde{y}) = |x - y|$ (the elements \tilde{x} and \tilde{y} are generated by Cauchy sequences of points in $]x, y[$ converging to the respective points x and y). Similarly, each boundary angle $\angle(x, y, z)$ naturally generates a triple $\tilde{x}, \tilde{y}, \tilde{z}$ of elements of the Hausdorff boundary $\partial_H U$ such that $p\tilde{x} = x$, $p\tilde{y} = y$, $p\tilde{z} = z$, $\rho_{\partial_H U, U}(\tilde{x}, \tilde{y}) = |x - y|$, and $\rho_{\partial_H U, U}(\tilde{z}, \tilde{y}) = |z - y|$. The same can be said about the boundary triangle $\Delta(x, y, z)$ for which we also have $\rho_{\partial_H U, U}(\tilde{x}, \tilde{z}) = |x - z|$.

Assume that $f : \partial_H U \rightarrow \partial_H V$ is a mapping, $x \in \partial U$, $\tilde{x} \in \partial_H U$, and $p\tilde{x} = x$ (usually, by default the point \tilde{x} is determined from the context, for example, as the endpoint of a boundary interval; see above). Now put $x' = pf(\tilde{x})$. Generally, given a point $x \in \text{cl } U$, we denote by x' the corresponding point $x' \in \text{cl } V$ when the correspondence is clear from the context.

Clearly, the intrinsic metric of U can be extended by continuity to $U \cup \partial_H U$. Thus, the value $\rho_{\partial_H U, U}(\tilde{x}, \tilde{y})$ is defined for every pair x, y in $U \cup \partial_H U$.

We need a series of lemmas; moreover, the main tools here are Lemmas 2.3 and 2.9.

Lemma 2.3 (on invariance of a boundary interval). *Suppose that $U, V \subset \mathbb{R}^n$ are domains and $f : \partial_H U \rightarrow \partial_H V$ is an isometry (in the relative metrics) of the Hausdorff boundaries of these domains. Suppose that $]x, y[$ is a boundary interval for the domain U . Then $]x', y'[$ is a boundary interval for V ; moreover, $|x' - y'| = |x - y|$.*

Some particular cases of this assertion were known earlier: for example, in the case when U is a bounded convex domain a similar assertion is contained in Lemma 4 of [2].

Proof of Lemma 2.3. Since f is an isometry, we have

$$\rho_{\partial_H V, V}(\tilde{x}, \tilde{y}) = |x - y|. \quad (2.1)$$

Take some Cauchy sequences $v_\nu \in V \rightarrow x'$ and $w_\nu \in V \rightarrow y'$ (with respect to the intrinsic metric of V) generating elements $f(\tilde{x})$ and $f(\tilde{y})$ of the Hausdorff boundary. (Existence of these sequences follows from the definition of Hausdorff boundary.) Now, take a sequence of curves $\gamma_\nu : [0, 1] \rightarrow V$ such that $\gamma_\nu(0) = v_\nu$, $\gamma_\nu(1) = w_\nu$, and $l(\gamma_\nu) \rightarrow \rho_{\partial_H V, V}(f(\tilde{x}), f(\tilde{y})) = |x - y|$. Without loss of generality we can also assume that the parameterizations

of γ_ν coincide to within factor with the natural parameterizations and the mappings γ_ν converge uniformly to the mapping $\gamma : [0, 1] \rightarrow \text{cl } V$ so that $\gamma(0) = x'$, $\gamma(1) = y'$, γ is a rectifiable curve, and

$$l(\gamma) \leq \rho_{\partial_H V, V}(f(\tilde{x}), f(\tilde{y})). \quad (2.2)$$

To complete the proof of Lemma 2.3, it suffices now to show that

$$\forall t \in]0, 1[\quad \gamma(t) \in V. \quad (2.3)$$

Indeed, if (2.3) is valid then, by the definition of the relative metric on the Hausdorff boundary, we have $l(\gamma) \geq \rho_{\partial_H V, V}(f(\tilde{x}), f(\tilde{y}))$. Combining this with (2.2), we obtain

$$l(\gamma) = \rho_{\partial_H V, V}(f(\tilde{x}), f(\tilde{y})). \quad (2.4)$$

From (2.3) and (2.4) we easily find that the curve γ is a line segment with endpoints x' and y' ; moreover, $]x', y'[\subset V$ and $|x' - y'| = \rho_{\partial_H V, V}(f(\tilde{x}), f(\tilde{y}))$. This together with (2.1) gives the claim of Lemma 2.3.

Thus, we are left with proving (2.3). Assume (2.3) false. Then there is $t_0 \in]0, 1[$ such that $\gamma(t_0) \in \partial V$. Put $r_\nu = l(\gamma_\nu([0, t_0]))$, $s_\nu = l(\gamma_\nu([t_0, 1]))$, and $R_\nu = \text{dist}(\gamma_\nu(t_0), \partial V)$. By the above assumptions, $R_\nu \rightarrow 0$ as $\nu \rightarrow \infty$, $\forall \nu = 1, 2, \dots$ ($B(\gamma_\nu(t_0), R_\nu) \subset V$ & $\exists A_\nu \in \text{cl } B(\gamma_\nu(t_0), R_\nu) \cap \partial V$), and

$$r_\nu + s_\nu \rightarrow |x - y| \quad \text{as } \nu \rightarrow \infty, \quad (2.5)$$

$$\exists \varepsilon_0 > 0 \quad \forall \nu \in \mathbb{N} \quad (r_\nu > \varepsilon_0 \text{ \& } s_\nu > \varepsilon_0). \quad (2.6)$$

By (2.5) and (2.6), we can assume without loss of generality that $r_\nu \rightarrow r > 0$, $s_\nu \rightarrow s > 0$, and

$$r + s = |x - y|. \quad (2.7)$$

If we take a sequence of points in the ball $B(\gamma_\nu(t_0), R_\nu)$ converging to A_ν then it generates some element of the Hausdorff boundary $\tilde{A}_\nu \in \partial_H V$. It is easy to see that $\limsup_{\nu \rightarrow \infty} \rho_{\partial_H V, V}(f(\tilde{x}), \tilde{A}_\nu) \leq r$ and $\limsup_{\nu \rightarrow \infty} \rho_{\partial_H V, V}(f(\tilde{y}), \tilde{A}_\nu) \leq s$. Denote $\tilde{B}_\nu = f^{-1}(\tilde{A}_\nu)$. Since the mapping f is an isometry, we find

$$\limsup_{\nu \rightarrow \infty} \rho_{\partial_H V, V}(\tilde{x}, \tilde{B}_\nu) \leq r, \quad (2.8)$$

$$\limsup_{\nu \rightarrow \infty} \rho_{\partial_H V, V}(\tilde{y}, \tilde{B}_\nu) \leq s. \quad (2.9)$$

Put $B_\nu = p\tilde{B}_\nu$. Then, by the definition of $\rho_{\partial_H U, U}$, the following are valid:

$$|x - B_\nu| \leq \rho_{\partial_H V, V}(\tilde{x}, \tilde{B}_\nu), \quad (2.10)$$

$$|y - B_\nu| \leq \rho_{\partial_H V, V}(\tilde{y}, \tilde{B}_\nu), \quad (2.11)$$

It follows from (2.7)-(2.11) and the triangle inequality that $B_\nu \rightarrow B = \frac{s}{|x-y|}x + \frac{r}{|x-y|}y$. Since $B_\nu \in \partial U$, we have $B \in \partial U$. However, the last fact contradicts the condition $]x, y[\subset U$ of Lemma 2.3. The contradiction completes the proof of Lemma 2.3.

In the lemmas below, we suppose unless the contrary is specified that U and V are domains in \mathbb{R}^2 and $f : \partial_H U \rightarrow \partial_H V$ is an isometry (in the relative metrics) of the Hausdorff boundaries of these domains.

Remark 2.1. With these definitions, the mapping $f : \partial_H U \rightarrow \partial_H V$ is an isometry of the Hausdorff boundaries if and only if the inverse mapping $f^{-1} : \partial_H V \rightarrow \partial_H U$ is an isometry of the Hausdorff boundaries. Therefore, the following lemmas remain valid with U and V interchanged.

Lemma 2.4 *Let $\angle(x, y, z)$ be a boundary angle for U . Then $|x' - y'| = |x - y|$, $|z' - y'| = |z - y|$, and $|x' - z'| \leq |x - z|$.*

We can express the gist of this lemma as follows: The mapping f takes boundary angles into the angles equal or less than the original (so far we do not claim that these angles are boundary angles).

Below we need one more denotation. Let $\angle(x, y, z)$ be a boundary angle for some domain Ω . Put $E = E(\angle(x, y, z), \Omega) = \text{conv}(\angle(x, y, z) \cap (\partial\Omega) \setminus \{y\})$. It follows from the definitions that $[x, z] \subset E$ and

$$\exists r_1 > 0 \quad B(y, r_1) \cap E = \emptyset. \quad (2.12)$$

Denote

$$\Gamma_{\angle(x, y, z)}^\Omega = \{w \in (\Omega) \cap (\partial E) : (y, w) \subset \Omega \setminus E\}. \quad (2.13)$$

Observe, in particular, that by construction

$$x, z \in \Gamma_{\angle(x, y, z)}^\Omega, \quad (2.14)$$

$$\forall w \in \Gamma_{\angle(x, y, z)}^\Omega \quad]y, w[\text{ is a boundary interval for } \Omega. \quad (2.15)$$

Proof of Lemma 2.4. Making parallel translations, if necessary, we can assume without loss of generality that

$$y = y' = 0 \in \mathbb{R}^2. \quad (2.16)$$

Let $\Phi = |\angle(x, 0, z)|$ and $E = E(\angle(x, 0, z), U)$ (see above). Construct an arc $\gamma : [0, \Phi] \rightarrow \partial E$, defining the point $\gamma(\varphi)$ by the following properties: $\gamma(\varphi) \in \angle(x, 0, z)$, $\gamma(\varphi) \in E$, $]0, \gamma(\varphi)[\cap E = \emptyset$, and $|\angle(x, 0, \gamma(\varphi))| = \varphi$. It is clear that $\gamma(0) = x$, $\gamma(\Phi) = z$, and γ is a convex arc in $E \cap \partial E$ joining x and z .

Denote $\Gamma = \gamma([0, 1])$, $\Gamma_1 = \Gamma \cap \partial U$, and $\Gamma_2 = \Gamma \setminus \Gamma_1$. From the above $\Gamma_1 = \Gamma_{\angle(x, 0, z)}^U$, whence, by (2.15),

$$\forall w \in \Gamma_1, \quad]0, w[\text{ is a boundary interval.} \quad (2.17)$$

Moreover, by construction, $\forall w \in \Gamma_2, \quad \exists w_1, w_2 \in \Gamma_1 \quad w \in]w_1, w_2[\subset U \cap \Gamma$.

Define $g : \Gamma_1 \rightarrow \partial V$ by the rule $g(w) = w'$ for $w \in \Gamma_1$. By Lemma 2.3 (see (2.16) and (2.17)),

$$\forall w \in \Gamma_1, \quad |g(w)| = |w|. \quad (2.18)$$

From the above constructions we also derive the equalities

$$x' = g(x), \quad z' = g(z). \quad (2.19)$$

From here and (2.18) we obtain in particular that

$$|x'| = |x|, \quad |z'| = |z|. \quad (2.20)$$

Since Γ is a convex arc and all its extreme³ points lie in Γ_1 , it is geometrically obvious that, for each $\varepsilon > 0$, there is a partition $0 < \varphi_0 < \varphi_1 < \dots < \varphi_N = \Phi$, $N = N(\varepsilon)$, such that $\forall \nu = 0, 1, \dots, N \quad \gamma(\varphi_\nu) \in \Gamma_1$,

$$\forall \nu = 0, 1, \dots, N-1 \quad \frac{l(\gamma([\varphi_\nu, \varphi_{\nu+1}]))}{|\gamma(\varphi_\nu) - \gamma(\varphi_{\nu+1})|} \leq 1 + \varepsilon. \quad (2.21)$$

Using the definition of Hausdorff metric, we find that

$$\forall \nu = 0, 1, \dots, N-1 \quad \rho_{U_H}(\widetilde{\gamma(\varphi_\nu)}, \widetilde{\gamma(\varphi_{\nu+1})}) \leq l(\gamma([\varphi_\nu, \varphi_{\nu+1}])). \quad (2.22)$$

From (2.21) and (2.22) and the definition of Hausdorff metric we conclude that

$$\begin{aligned} \forall \nu = 0, 1, \dots, N-1 \quad |g(\gamma(\varphi_\nu)) - g(\gamma(\varphi_{\nu+1}))| &\leq \rho_{V_H}(f(\widetilde{\gamma(\varphi_\nu)}), f(\widetilde{\gamma(\varphi_{\nu+1})})) = \\ \rho_{U_H}(\widetilde{\gamma(\varphi_\nu)}, \widetilde{\gamma(\varphi_{\nu+1})}) &\leq l(\gamma([\varphi_\nu, \varphi_{\nu+1}])) \leq (1 + \varepsilon)|\gamma(\varphi_\nu) - \gamma(\varphi_{\nu+1})|. \end{aligned} \quad (2.23)$$

³An extreme point of a set F is a point $A \in F$ such that there is no pair of points $B, C \in F$ different from A and a number $p \in]0, 1[$ for which $A = pB + (1-p)C$.

From the arbitrariness of $\varepsilon > 0$, (2.18), (2.23), and obvious geometric arguments, we infer that $|g(z) - g(x)| \leq |x - z|$ which together with (2.19) and (2.20) gives the claim of Lemma 2.4.

Introduce one more definition. Take $\tilde{x}, \tilde{y} \in \partial_H U$. We say that a rectifiable curve $\gamma : [0, L] \rightarrow \text{cl } U$ is an H -shortest curve between \tilde{x} and \tilde{y} if there is a sequence of rectifiable naturally parameterized curves $\gamma_\nu : [0, L_\nu] \rightarrow U$ such that $L_\nu \rightarrow L = |\tilde{x} - \tilde{y}|$, $\sup_{t \in [0, L]} |\gamma_\nu(t \frac{L_\nu}{L}) - \gamma(t)| \rightarrow 0$, $\rho_{\partial_H U, U}(\gamma_\nu(0), \tilde{x}) \rightarrow 0$, and $\rho_{\partial_H U, U}(\gamma_\nu(L_\nu), \tilde{y}) \rightarrow 0$ as $\nu \rightarrow \infty$.

Lemma 2.5 *For every pair $\tilde{x}, \tilde{y} \in \partial_H U$, there is an H -shortest curve. Moreover, every H -shortest curve $\gamma : [0, L] \rightarrow \text{cl } U$ possesses the following properties:*

(H-i) *Given a pair of numbers $\varphi_1, \varphi_2 \in [0, L]$, if*

$$\gamma(\varphi_1), \gamma(\varphi_2) \in \partial U \text{ and } \gamma(\varphi) \in U \text{ for all } \varphi \in]\varphi_1, \varphi_2[\quad (2.24)$$

then $\gamma(\varphi) \in]\gamma(\varphi_1), \gamma(\varphi_2)[$ for all $\varphi \in]\varphi_1, \varphi_2[$.

(H-ii) *If $\gamma(\varphi) \in U$ then*

$$l(\gamma([0, \varphi])) \leq \rho_{U_H}(\tilde{x}, \gamma(\varphi)) = \varphi, \quad l(\gamma([\varphi, L])) \leq \rho_{U_H}(\tilde{y}, \gamma(\varphi)) = L - \varphi.$$

The assertion (H-i) means that the intersection of an H -shortest curve with U consists of boundary intervals.

Proof of Lemma 2.5 is simple and therefore omitted.

Lemma 2.6 *Take $\tilde{x}, \tilde{y} \in \partial_H U$ and let $\gamma : [0, L] \rightarrow \text{cl } V$ be an H -shortest curve between \tilde{x} and \tilde{y} . Then there is an H -shortest curve $\gamma_V : [0, L] \rightarrow \text{cl } V$ between $f(\tilde{x})$ and $f(\tilde{y})$ with the following properties:*

(H-iii) $\forall \varphi \in]0, L[\quad (\gamma(\varphi) \in U \Leftrightarrow \gamma_V(\varphi) \in V).$

(H-iv) *For every pair of numbers $\varphi_1, \varphi_2 \in [0, L]$ satisfying (2.24) we have: $f(\gamma(\varphi_1)) = \gamma_V(\varphi_1)$ and $f(\gamma(\varphi_2)) = \gamma_V(\varphi_2)$.*

The proof relies on the corresponding definitions and Lemmas 2.2, 2.3, 2.5 and is carried out by the standard arguments of calculus; therefore, we omit it.

Lemma 2.7 *Let $\Delta(x, y, z)$ be a boundary triangle for U . Then $\Delta(x', y', z')$ is a boundary triangle for V ; moreover,*

$$|x' - y'| = |x - y|, \quad |z' - y'| = |z - y|, \quad |x' - z'| = |x - z|.$$

We can express the essence of this lemma as follows: The mapping f takes boundary triangles into equal boundary triangles.

Proof. From the definition of a boundary triangle we obtain in particular

$$\text{Int } \Delta(x, y, z) \subset U. \quad (2.25)$$

It follows from Lemma 2.3 that $x', y', z' \in \partial V$ and

$$]x', y'[\cup]y', z'[\cup]x', z'[\subset V, \quad (2.26)$$

$$|x' - y'| = |x - y|, \quad |z' - y'| = |z - y|, \quad |x' - z'| = |x - z|. \quad (2.27)$$

It remains to prove that

$$\text{Int } \Delta(x', y', z') \subset V. \quad (2.28)$$

Assume that some point w satisfies the conditions $w \in \partial U$, $]y, w[\subset U$, and $]y, w[\cap]x, z[\neq \emptyset$. Then it is geometrically obvious that $\angle(x, y, w)$ and $\angle(w, y, z)$ are boundary angles. Then, by Lemmas 2.3 and 2.4, $]y', w'[\subset V$, $|w' - x'| \leq |w - x|$, $|w' - z'| \leq |w - z|$, and

$$|w' - y'| = |w - y|. \quad (2.29)$$

Hence, using (2.27), we conclude that

$$|w' - x'| = |w - x|, \quad |w' - z'| = |w - z|. \quad (2.30)$$

Equalities (2.27), (2.29), and (2.30) demonstrate that the figure consisting of the four points x, y, z , and w is Euclideanly isometric to the figure consisting of x', y', z' , and w' .

We have thus proven the implication

$$(w \in \partial U,]y, w[\subset U,]y, w[\cap]x, z[\neq \emptyset) \Rightarrow (]y', w'[\subset V, |w' - y'| = |w - y|, |w' - x'| = |w - x|, |w' - z'| = |w - z|). \quad (2.31)$$

Similarly, we can prove the implications

$$(w \in \partial U,]x, w[\subset U,]x, w[\cap]y, z[\neq \emptyset) \Rightarrow (]x', w'[\subset V, |w' - y'| = |w - y|, |w' - x'| = |w - x|, |w' - z'| = |w - z|),$$

$$(w \in \partial U,]z, w[\subset U,]z, w[\cap]x, y[\neq \emptyset) \Rightarrow (]z', w'[\subset V, |w' - y'| = |w - y|, |w' - x'| = |w - x|, |w' - z'| = |w - z|).$$

Now, consider the case when

$$\exists r > 0 \quad B(y', r) \cap (\text{Int } \Delta(x', y', z')) \cap \partial V = \emptyset. \quad (2.32)$$

Then the desired equality (2.28) is very easy to prove. Indeed, if (2.32) is valid but (2.28) is violated then there is a point $v \in (\text{Int } \Delta(x', y', z')) \cap \partial V$ such that $\angle(x', y', v)$ and $\angle(v, y', z')$ are boundary angles. Take the corresponding element $\tilde{v} \in \partial_H V$, $p\tilde{v} = v$, and consider $u = pf^{-1}(\tilde{v}) \in \partial U$. Using the same method as in the proof of (2.31), we can show that the figure consisting of the four points x' , y' , z' , and v is Euclideanly isometric to the figure consisting of the points x , y , z , and u . But then $u \in (\text{Int } \Delta(x, y, z)) \cap \partial U$ which contradicts (2.25).

Thus, it remains to consider the case when (2.32) is false, i.e., when

$$\forall r > 0 \quad B(y', r) \cap (\text{Int } \Delta(x', y', z')) \cap \partial V \neq \emptyset. \quad (2.33)$$

For the same reasons, without loss of generality we can additionally assume that

$$\forall r > 0 \quad B(x', r) \cap (\text{Int } \Delta(x', y', z')) \cap \partial V \neq \emptyset, \quad (2.34)$$

$$\forall r > 0 \quad B(z', r) \cap (\text{Int } \Delta(x', y', z')) \cap \partial V \neq \emptyset. \quad (2.35)$$

Suppose now that at least one of the following two assertions is valid:

$$\exists w_0 \in]x, z[\quad \forall w_1 \in]w_0, z[\quad \exists w \in \partial U \quad w_1 \in]y, w[\subset U \quad (2.36)$$

or

$$\exists w_0 \in]x, z[\quad \forall w_1 \in]x, w_0[\quad \exists w \in \partial U \quad w_1 \in]y, w[\subset U. \quad (2.37)$$

It follows from (2.31) that

$$\exists w'_0 \in]x', z'[\quad \forall w'_1 \in]w'_0, z'[\quad \exists w' \in \partial V \quad w'_1 \in]y', w'[\subset V \quad (2.38)$$

or

$$\exists w'_0 \in]x', z'[\quad \forall w'_1 \in]x', w'_0[\quad \exists w' \in \partial V \quad w'_1 \in]y', w'[\subset V. \quad (2.39)$$

If (2.38) holds, then in particular $\exists w'_0 \in]x', z'[\quad \forall w'_1 \in]w'_0, z'[\quad]y', w'_1[\subset V$ which contradicts (2.35). If (2.39) holds then we similarly obtain a contradiction with (2.34).

We have thus proven that (2.34) and (2.35) imply that none of the assertions (2.36) and (2.37) is valid, i.e.,

$$\forall w_0 \in]x, z[\quad \exists w_1 \in]w_0, z[\quad \{w \in \mathbb{R}^2 : w - y = \tau(w_1 - y), \tau > 0\} \subset U;$$

$$\forall w_0 \in]x, z[\quad \exists w_1 \in]x, w_0[\quad \{w \in \mathbb{R}^2 : w - y = \tau(w_1 - y), \tau > 0\} \subset U.$$

It is geometrically obvious from here, (2.25), and property (H-i) that for every point $\tilde{u} \in \partial_H U$ such that $u = p\tilde{u} \in \text{Int } \angle(x, y, z)$ and for each H -shortest curve $\gamma : [0, L] \rightarrow \mathbb{R}^2$ joining \tilde{y} and \tilde{u} , there exist $w \in \angle(x, y, z) \cap \partial U \setminus \Delta(x, y, z)$ and $\alpha_0 \in]0, L]$ such that $\gamma(\alpha_0) = w$ and $\gamma(\tau) \in]y, w[\subset U$ for $\tau \in]0, \alpha_0[$. On the other hand, it is geometrically obvious that if an H -shortest curve joins the vertex $f(\tilde{y})$ with a point of $\text{Int } \Delta(x' y' z')$ then the H -shortest curve cannot leave the triangle $\Delta(x' y' z')$. Therefore, from the above, Lemma 2.6, and (2.31) we obtain the implications

$$\forall \tilde{v} \in \partial_H V \quad (v = p\tilde{v} \in \text{Int } \Delta(x', y', z') \Rightarrow u = pf^{-1}(\tilde{v}) \notin \text{Int } \angle(x, y, z)). \quad (2.40)$$

Thus, from (2.34) and (2.35) we arrive at (2.40). Similarly, from (2.33) and (2.35) we can also obtain two more implications

$$\forall \tilde{v} \in \partial_H V \quad (v = p\tilde{v} \in \text{Int } \Delta(x', y', z') \Rightarrow u = pf^{-1}(\tilde{v}) \notin \text{Int } \angle(z, x, y)); \quad (2.41)$$

$$\forall \tilde{v} \in \partial_H V \quad (v = p\tilde{v} \in \text{Int } \Delta(x', y', z') \Rightarrow u = pf^{-1}(\tilde{v}) \notin \text{Int } \angle(y, z, x)). \quad (2.42)$$

We can assume without loss of generality that

$$|y' - z'|, |x' - y'| \leq |x' - z'|. \quad (2.43)$$

From (2.26), (2.34), and (2.35) we easily find now that there exist sequences

$$\text{Int } \Delta(x', y', z') \ni v_\nu^1 \rightarrow x', \quad \text{Int } \Delta(x', y', z') \ni v_\nu^2 \rightarrow z' \quad (2.44)$$

such that $v_\nu^1, v_\nu^2 \in \partial V$ and $]v_\nu^1, v_\nu^2[\subset V$ (i.e., $]v_\nu^1, v_\nu^2[$ is a boundary interval for V) and the following hold:

$$\rho_{\partial_H V, V}(\tilde{v}_\nu^1, f(\tilde{x})) \rightarrow 0, \quad \rho_{\partial_H V, V}(\tilde{v}_\nu^2, f(\tilde{z})) \rightarrow 0.$$

From (2.43) and (2.44) we obtain $|v_\nu^1 - v_\nu^2| < |x' - z'|$. Put $u_\nu^1 = pf^{-1}(\tilde{v}_\nu^1)$ and $u_\nu^2 = pf^{-1}(\tilde{v}_\nu^2)$. Then from Lemma 2.3 and the above assumptions we find that $u_\nu^1, u_\nu^2 \in \partial U$, $]u_\nu^1, u_\nu^2[\subset U$, and

$$u_\nu^1 \rightarrow x, \quad u_\nu^2 \rightarrow z, \quad |u_\nu^1 - u_\nu^2| = |v_\nu^1 - v_\nu^2| < |x' - z'| = |x - z|. \quad (2.45)$$

Moreover, (2.40)-(2.42) and the previous computations imply that

$$u_\nu^1, u_\nu^2 \notin (\text{Int } \Delta(x, y, z)) \cup (\text{Int } \Delta(z, x, y)) \cup (\text{Int } \Delta(y, z, x)) \cup \Delta(x, y, z). \quad (2.46)$$

But (2.45) and (2.46) contradict each other. The contradiction completes the proof of Lemma 2.7.

Lemma 2.8 *Let $\angle(x, y, z)$ be a boundary angle for U . Then $\angle(x', y', z')$ is a boundary angle for V .*

We can express the essence of this lemma as follows: The mapping f takes boundary angles into boundary angles.

Proof. It follows from Lemmas 2.3 and 2.4 that $x', y', z' \in \partial V$, $]x', y'[\cup]y', z'[\subset V$, $|x' - y'| = |x - y|$, $|z' - y'| = |z - y|$, and $|x' - z'| \leq |x - z|$. These formulas and the fact that x, y , and z do not lie on one straight line imply that x', y' , and z' do not lie on one straight line, too.

It remains to prove that

$$\exists r > 0 \quad B(y', r) \cap \angle(x', y', z') \setminus \{y'\} \subset V. \quad (2.47)$$

Making parallel translations, if necessary, we can assume without loss of generality that $y = y' = 0 \in \mathbb{R}^2$. We proceed firstly as in the proof of Lemma 2.4: repeat all arguments of the proof of Lemma 2.4 up to (2.20). Consider the above-constructed mapping $g : \Gamma_1 \rightarrow \mathbb{R}^2$. By construction,

$$\forall w \in \Gamma_1, \quad]0, g(w)[\subset V. \quad (2.48)$$

Extend g to the whole set Γ as follows: Let $w \in \Gamma \setminus \Gamma_1 (= \Gamma_2)$. It is geometrically obvious that $\exists w_1, w_2 \in \Gamma_1 \quad w \in]w_1, w_2[\subset U$. It is also geometrically obvious that the points w_1 and w_2 from the previous formula are determined uniquely; moreover,

$$]w_1, w_2[\subset U \cap \Gamma, \quad (2.49)$$

$$\Delta(0, w_1, w_2) \text{ is a boundary triangle for } U. \quad (2.50)$$

In this case put $g(w) = sg(w_1) + (1-s)g(w_2)$, where $s \in]0, 1[$ is determined from the equality $w = sw_1 + (1-s)w_2$.

From the definition of g and (2.18) we immediately find that

$$\forall w \in \Gamma, \quad |g(w)| = |w|. \quad (2.51)$$

Define the curve $\gamma_V : [0, \Phi] \rightarrow \mathbb{R}^2$ by the rule $\gamma_V(\varphi) = g(\gamma(\varphi))$ for $\varphi \in [0, \Phi]$, where $\gamma(\varphi)$ is defined in the proof of Lemma 2.4. By construction, γ_V is continuous, and from (2.19) and (2.51) we obtain

$$x' = \gamma_V(0), \quad z' = \gamma_V(\Phi) \quad (2.52)$$

and $\forall \varphi \in [0, \Phi] \quad |\gamma_V(\varphi)| = |\gamma(\varphi)|$. Using (2.12) and the construction of $\gamma(\varphi)$, from the last identity we conclude that there exists $r_1 > 0$ such that

$$\forall \varphi \in [0, \Phi], \quad |\gamma_V(\varphi)| \geq r_1. \quad (2.53)$$

By (2.48),

$$\forall \varphi \in [0, \Phi], \quad (\gamma(\varphi) \in \Gamma_1 \Rightarrow]0, \gamma_V(\varphi)[\subset V). \quad (2.54)$$

Now, by construction (in particular, see (2.49) and (2.50)) and Lemma 2.7,

$$\forall \varphi \in [0, \Phi], \quad (\gamma(\varphi) \in \Gamma_2 \Rightarrow]0, \gamma_V(\varphi)[\subset V). \quad (2.55)$$

Combining (2.54) and (2.55), we finally obtain the desired inclusions

$$\forall \varphi \in [0, \Phi], \quad]0, \gamma_V(\varphi)[\subset V. \quad (2.56)$$

Now, (2.56), (2.53), (2.52), and continuity of γ_V yield the desired relation (2.47). Lemma 2.8 is proven.

Lemma 2.9 *Let $\angle(x, y, z)$ be a boundary angle for U . Then $\angle(x', y', z')$ is a boundary angle for V ; moreover, $|x' - y'| = |x - y|$, $|z' - y'| = |z - y|$, and $|x' - z'| = |x - z|$.*

We can express the essence of this lemma as follows: The mapping f takes boundary angles into equal boundary angles.

Proof. We have to apply Lemmas 2.4 and 2.8 twice.

In view of (2.14), the technical assertion below contains Lemma 2.9 as a particular case:

Lemma 2.10 *Let $\angle(x, y, z)$ be a boundary angle for U . Then*

$$\Gamma_{\angle(x', y', z')}^V = \{w' : w \in \Gamma_{\angle(x, y, z)}^U\}; \quad (2.57)$$

moreover,

$$|w'_1 - w'_2| = |w_1 - w_2|, \quad \forall w_1, w_2 \in \Gamma_{\angle(x, y, z)}^U \cup \{y\}. \quad (2.58)$$

Proof. By Lemma 2.9, $\angle(x', y', z')$ is a boundary angle for V which is Euclideanly isometric to the angle $\angle(x, y, z)$. It is geometrically obvious now that, for every $w \in \Gamma_{\angle(x, y, z)}^U \setminus \{x, z\}$, $\angle(x, y, w)$ and $\angle(w, y, z)$ are boundary angles for U . Similarly, for every $v \in \Gamma_{\angle(x', y', z')}^V \setminus \{x', z'\}$, $\angle(x', y', v)$ and $\angle(v, y', z')$ are boundary angles for V . Now, the desired assertions (2.57) and (2.58) are obtained from the last two propositions, Lemma 2.9, and the following fact:

(P₁) *The position of a point on a plane is determined uniquely from its distances to three points not lying on one straight line.*

We drop the remaining technical details which are plain.

Lemma 2.11 *Suppose that $x_i \in \partial U$, $i = 1, \dots, m$, are such that $\angle(x_i, x_{i+1}, x_{i+2})$ constitute boundary angles for U for all $i = 1, \dots, m - 2$. Then $|x'_i - x'_j| = |x_i - x_j|$ for all $i, j = 1, \dots, m$.*

Proof. For $m = 3$ the assertion of Lemma 2.11 coincides with the assertion of Lemma 2.9. Now, it clearly suffices to prove Lemma 2.11 in the case $m = 4$, since for $m > 4$ the assertion of Lemma 2.11 is derived by induction with use of the simple fact (P_1) (see above). Thus, we assume below that $m = 4$.

It follows from Lemma 2.9 and the definition of boundary angles that

$$|x'_i - x'_j| = |x_i - x_j|, \quad i, j = 1, 2, 3, \quad (2.59)$$

$$|x'_i - x'_j| = |x_i - x_j|, \quad i, j = 2, 3, 4, \quad (2.60)$$

$$x'_1, x'_2, \text{ and } x'_3 \text{ are not collinear,} \quad (2.61)$$

$$x'_2, x'_3, \text{ and } x'_4 \text{ are not collinear.} \quad (2.62)$$

It suffices to prove the only equality

$$|x'_1 - x'_4| = |x_1 - x_4|. \quad (2.63)$$

Making isometric transformations, if necessary, and using (2.59), we can assume without loss of generality that

$$x'_i = x_i, \quad i = 1, 2, 3. \quad (2.64)$$

Then, by (2.59)-(2.62), the desired equality (2.63) is equivalent to $x'_4 = x_4$. Assume that the latter is false. Then, by (2.59)-(2.62),

$$x'_4 \text{ is the reflection of } x_4 \text{ in the straight line } x_2x_3. \quad (2.65)$$

It follows from (2.64) and (2.58) that

$$w' = w, \quad \text{for all } w \in \Gamma_{\angle(x_1, x_2, x_3)}^U. \quad (2.66)$$

From (2.65), (2.64), and (2.58) we find that

$$w' \text{ is the reflection of } w \text{ in the straight line } x_2x_3 \text{ for all } w \in \Gamma_{\angle(x_2, x_3, x_4)}^U. \quad (2.67)$$

Then, by interchanging the domains U and V , if necessary, we can assume without loss of generality that x_1 and x_4 lie on one side of the straight line x_2x_3 and x'_1 and x'_4 lie on the different sides of the straight line $x'_2x'_3$. It is

geometrically obvious now that in our situation we deal with at least one of the following three possibilities:⁴

- (i) there is a boundary angle $\angle(x_1, x_2, u)$ for U , where $u \in \Gamma_{\angle(x_2, x_3, x_4)}^U \setminus \{x_2\}$;
- (ii) there is a boundary angle $\angle(x_4, x_3, u)$ for U , where $u \in \Gamma_{\angle(x_1, x_2, x_3)}^U \setminus \{x_3\}$;
- (iii) there is a boundary interval $]v_1, v_2[$ for V , joining $v_1 \in \Gamma_{\angle(x'_1, x'_2, x'_3)}^V \setminus \{x'_3\}$ and $v_2 \in \Gamma_{\angle(x'_2, x'_3, x'_4)}^V \setminus \{x'_2\}$.

By Lemmas 2.3 and 2.10, (iii) is equivalent to the following:

- (iv) there is a boundary interval $]u_1, u_2[$ (for U) joining $u_1 \in \Gamma_{\angle(x_1, x_2, x_3)}^U \setminus \{x_3\}$ and $u_2 \in \Gamma_{\angle(x_2, x_3, x_4)}^U \setminus \{x_2\}$.

Therefore, we can assume that at least one of the three assertions is always valid: (i), (ii), or (iv). But, by Lemmas 2.3 and 2.9, each of these assertions implies the following:

- (v) there exist⁵ points $u_1 \in \Gamma_{\angle(x_1, x_2, x_3)}^U \setminus \{x_3\}$ and $u_2 \in \Gamma_{\angle(x_2, x_3, x_4)}^U \setminus \{x_2\}$ such that

$$|u'_2 - u'_1| = |u_2 - u_1|. \quad (2.68)$$

It is geometrically obvious that

$$u_1 \text{ does not lie on the straight line } x_2x_3, \quad (2.69)$$

$$u_2 \text{ does not lie on the straight line } x_2x_3. \quad (2.70)$$

From Lemma 2.10 (see (2.14) and (2.58)) and the membership $u_2 \in \Gamma_{\angle(x_2, x_3, x_4)}^U$ we also obtain the equalities

$$|u'_2 - x'_2| = |u_2 - x_2|, \quad (2.71)$$

$$|u'_2 - x'_3| = |u_2 - x_3|. \quad (2.72)$$

Using (2.64) and (2.66), we can rewrite (2.68), (2.71), and (2.72) as $|u'_2 - u_1| = |u_2 - u_1|$, $|u'_2 - x_2| = |u_2 - x_2|$, and $|u'_2 - x_3| = |u_2 - x_3|$. Then from (2.69) and property (P_1) we find that $u'_2 = u_2$. But the last equality contradicts (2.67) and (2.70). The contradiction completes the proof of Lemma 2.11.

⁴ Assertion (iii) is valid if $x'_2 \in \text{cl}(\Gamma_{\angle(x'_2, x'_3, x'_4)}^V \setminus \{x'_2\})$ and $x'_3 \in \text{cl}(\Gamma_{\angle(x'_1, x'_2, x'_3)}^V \setminus \{x'_3\})$, and if at least one of these memberships is violated then one of the assertions (i) and (ii) is valid.

⁵The points u_1 and u_2 from (v) are determined as follows: if (iv) is valid then they coincide with the points in (iv) with the same names; if (i) is valid then $u_1 = x_1$ and $u_2 = u$, where u is defined in (i); if (ii) is valid then $u_1 = u$ and $u_2 = x_4$, where u is defined in (ii).

Lemma 2.12 *Suppose that $x_1, x_2, x_3 \in \partial U$ and $z \in]x_1, x_2[$ are such that $]x_1, x_2[$ is a boundary interval and $]x_3, z] \subset U$, $z = \tau x_1 + (1 - \tau)x_2 \in]x_1, x_2[$, $\tau \in]0, 1[$. Put $z' = \tau x'_1 + (1 - \tau)x'_2$ and $x'_3 = pf(\tilde{x}_3)$, where \tilde{x}_3 denotes the element of the Hausdorff boundary $\partial_H U$ generated by a Cauchy sequence of points of the interval $]x_3, z[$ converging to x_3 . Then $|x'_i - x'_j| = |x_i - x_j|$ for all $i, j = 1, \dots, 3$, $]x'_3, z'] \subset V$, and $f(\tilde{x}_3)$ coincides with the element of the Hausdorff boundary $\partial_H V$ generated by a Cauchy sequence of points of the interval $]x'_3, z'[$ converging to x'_3 .*

Proof. If x_1, x_2 , and x_3 are collinear, then x_3 coincides with x_1 or x_2 and there is nothing to prove. Henceforth we assume that x_1, x_2 , and x_3 are not collinear. Put $\Gamma_1 = \Gamma_{\angle(x_1, z, x_3)}^U$ and $\Gamma_2 = \Gamma_{\angle(x_2, z, x_3)}^U$, where these sets are defined as in the case of boundary angles (see (2.13)). It is geometrically obvious that there is a finite or infinite sequence of points u_ν , $\nu \in \mathbb{Z} \cap]\alpha, \omega[$, $-\infty \leq \alpha < \omega \leq +\infty$, such that $u_\nu \in \Gamma_1$ for odd ν and $u_\nu \in \Gamma_2$ for even ν ; moreover,

$\angle(u_\nu, u_{\nu+1}, u_{\nu+2})$ is a boundary angle for $\nu \in \mathbb{Z} \cap]\alpha, \omega - 2[$,

if $\omega = +\infty$, then $u_\nu \rightarrow x_3$ as $\nu \rightarrow +\infty$,

if $\omega < +\infty$, then $u_{\omega-1} = x_3$,

if $\alpha = -\infty$, then $u_{2k+1} \rightarrow x_1$ and $u_{2k} \rightarrow x_2$ as $k \rightarrow -\infty$,

if $\alpha > -\infty$, then $\{u_{\alpha+1}, u_{\alpha+2}\} = \{x_1, x_2\}$.

Now, the assertion of Lemma 2.12 easily follows from Lemmas 2.9-2.11 and obvious geometric arguments. We skip the details of computations in view of their simplicity.

Remark 2.2. Actually, Lemma 2.10 and the method of the proof of Lemma 2.12 yield the following stronger assertion:

$$\Gamma_{\angle(x'_1, z', x'_3)}^V = \{w' : w \in \Gamma_{\angle(x_1, z, x_3)}^U\}, \quad \Gamma_{\angle(x'_2, z', x'_3)}^V = \{w' : w \in \Gamma_{\angle(x_2, z, x_3)}^U\};$$

moreover,

$$|w'_1 - w'_2| = |w_1 - w_2| \quad \text{for all } w_1, w_2 \in \Gamma_{\angle(x_1, z, x_3)}^U \cup \Gamma_{\angle(x_2, z, x_3)}^U \cup \{z\}.$$

Proof of Item (I) of Theorem 1.1. Necessity easily follows from Lemmas 2.12 and 2.3, while sufficiency is obvious.

Remark 2.3. In view of Item (I) of Theorem 1.1, we will suppose unless the contrary is specified that there is no straight line containing ∂U or ∂V .

Lemma 2.13 *Suppose that $x_1, x_2, y_1, y_2 \in \partial U$ and $z_1, z_2 \in U$ are such that $]x_1, y_1[$ and $]x_2, y_2[$ are boundary intervals, $x_1 = \tau_1 x_1 + (1 - \tau_1) y_1 \in]x_1, y_1[$, and $x_2 = \tau_2 x_2 + (1 - \tau_2) y_2 \in]x_2, y_2[$, $\tau_1, \tau_2 \in]0, 1[$; moreover, $[x_1, x_2] \in U$. Put $x'_1 := \tau_1 x'_1 + (1 - \tau_1) y'_1$ and $x'_2 := \tau_2 x'_2 + (1 - \tau_2) y'_2$. Then $[z'_1, z'_2] \in V$ and the quadrangle $x_1 y_1 y_2 x_2$ is Euclideanly isometric to the quadrangle $x'_1 y'_1 y'_2 x'_2$.*

Proof. If $]x_1, y_1[\cap]x_2, y_2[\neq \emptyset$, then the assertion of Lemma 2.13 is easy from Lemma 2.12 and Remark 2.2. Now, consider the situation when $]x_1, y_1[\cap]x_2, y_2[= \emptyset$. We assume without loss of generality that x_1 and x_2 lie on one side of the straight line $x_1 x_2$. Consider the quadrangle G_x with vertices $z_1 x_1 x_2 z_2$. Put $E_x = \{w : \text{there is a collection of points } w_i \in \partial U, i = 1, \dots, k, \text{ such that } w \in \text{conv}(w_1, \dots, w_k) \subset G_x\}$. By construction, $[x_1, x_2] \subset E_x$. Put

$$\Gamma_x^U(x_1, y_1, x_2, y_2, z_1, z_2) = \{w \in (\partial U) \cap (\partial E_x) : \exists z \in [z_1, z_2] \]z, w[\subset U \setminus E_x\}.$$

By construction, we find also that $x_1, x_2 \in \Gamma_x^U(x_1, y_1, x_2, y_2, z_1, z_2)$. Similarly, consider the quadrangle G_y with vertices $z_1 y_1 y_2 z_2$. Put

$$E_y = \{w : \text{there is a collection of points } w_i \in \partial U, i = 1, \dots, k, \text{ such that } w \in \text{conv}(w_1, \dots, w_k) \subset G_y\}.$$

Denote

$$\Gamma_y^U(x_1, y_1, x_2, y_2, z_1, z_2) = \{w \in (\partial U) \cap (\partial E_y) : \exists z \in [z_1, z_2] \ }z, w[\subset U \setminus E_y\}.$$

By construction, $y_1, y_2 \in \Gamma_y^U(x_1, y_1, x_2, y_2, z_1, z_2)$. It is geometrically obvious that there is a finite or infinite sequence of points u_ν , $\nu \in \mathbb{Z} \cap]\alpha, \omega[$, $-\infty \leq \alpha < \omega \leq +\infty$, such that $u_\nu \in \Gamma_x^U(x_1, y_1, x_2, y_2, z_1, z_2)$ for odd ν and $u_\nu \in \Gamma_y^U(x_1, y_1, x_2, y_2, z_1, z_2)$ for even ν ; moreover,

$$\angle(u_\nu, u_{\nu+1}, u_{\nu+2}) \text{ is a boundary angle for } \nu \in \mathbb{Z} \cap]\alpha, \omega - 2[$$

$$\text{if } \omega = +\infty, \text{ then } u_{2k+1} \rightarrow x_2 \text{ and } u_{2k} \rightarrow y_2 \text{ as } k \rightarrow +\infty,$$

$$\text{if } \omega < +\infty, \text{ then } \{u_{\omega-1}, u_{\omega-2}\} = \{x_1, y_1\},$$

$$\text{if } \alpha = -\infty, \text{ then } u_{2k+1} \rightarrow x_1 \text{ and } u_{2k} \rightarrow y_1 \text{ as } k \rightarrow -\infty,$$

$$\text{if } \alpha > -\infty, \text{ then } \{u_{\alpha+1}, u_{\alpha+2}\} = \{x_2, y_2\}.$$

Now, the assertion of Lemma 2.13 is easy from Lemmas 2.9-2.11 and obvious geometric arguments. We skip the details of computations in view of their simplicity.

Remark 2.4. From Lemmas 2.10 and 2.11 and the method of the proof of Lemma 2.13 we derive the equalities

$$\begin{aligned}\Gamma_x^V(x'_1, y'_1, x'_2, y'_2, z'_1, z'_2) &= \{w' : w \in \Gamma_x^U(x_1, y_1, x_2, y_2, z_1, z_2)\}, \\ \Gamma_y^V(x'_1, y'_1, x'_2, y'_2, z'_1, z'_2) &= \{w' : w \in \Gamma_y^U(x_1, y_1, x_2, y_2, z_1, z_2)\}, \\ \forall w_1, w_2 \in \Gamma_x^U(x_1, y_1, x_2, y_2, z_1, z_2) \cup \Gamma_y^U(x_1, y_1, x_2, y_2, z_1, z_2) \quad &|w'_1 - w'_2| = |w_1 - w_2|.\end{aligned}$$

We say that boundary intervals $]x_1, y_1[$ and $]x_2, y_2[$ (with respect to U) are U -joinable if there exist points $x_1, x_2 \in U$ such that the conditions of Lemma 2.13 are satisfied.

Lemma 2.14 *Let $]x_i, y_i[$, $i = 1, \dots, m$, be boundary intervals (with respect to U). Suppose that the intervals $]x_i, y_i[$ and $]x_{i+1}, y_{i+1}[$ are U -joinable for all $i = 1, \dots, m-1$. Then the boundary intervals $]x'_i, y'_i[$ and $]x'_{i+1}, y'_{i+1}[$ are V -joinable for all $i = 1, \dots, m-1$; moreover,*

$$|x'_i - x'_j| = |x_i - x_j|, \quad |x'_i - y'_j| = |x_i - y_j|, \quad |y'_i - y'_j| = |y_i - y_j|, \quad i, j = 1, \dots, m. \quad (2.73)$$

Proof. For $m = 2$ the assertion of Lemma 2.14 coincides with that of Lemma 2.13. Now, it suffices clearly to prove Lemma 2.14 in the case $m = 3$, since for $m > 3$ the assertion of Lemma 2.14 is derived by induction on using the simple fact (P_1) (see above). Thus, we assume below that $m = 3$.

By the definition of U -joinable intervals, there exist $z_1^+, z_2^-, z_2^+, z_3^- \in U$ such that $z_i^+ = \tau_i^+ x_i + (1 - \tau_i^+) y_i \in]x_i, y_i[$, $z_i^- = \tau_i^- x_i + (1 - \tau_i^-) y_i \in]x_i, y_i[$, and $[z_i^+, z_{i+1}^-] \subset U$. Denote $z_i'^+ = \tau_i^+ x'_i + (1 - \tau_i^+) y'_i$ and $z_i'^- = \tau_i^- x'_i + (1 - \tau_i^-) y'_i$. Then Lemmas 2.13 and 2.3 immediately imply that $[z_i'^+, z_{i+1}'^-] \subset V$, $i = 1, 2$,

$$|x'_i - x'_j| = |x_i - x_j|, \quad |x'_i - y'_j| = |x_i - y_j|, \quad |y'_i - y'_j| = |y_i - y_j|, \quad i, j = 1, 2, \quad (2.74)$$

$$|x'_i - x'_j| = |x_i - x_j|, \quad |x'_i - y'_j| = |x_i - y_j|, \quad |y'_i - y'_j| = |y_i - y_j|, \quad i, j = 2, 3. \quad (2.75)$$

Making isometric transformations, if necessary, and using (2.74), we can assume without loss of generality that

$$x'_i = x_i, \quad y'_i = y_i, \quad i = 1, 2. \quad (2.76)$$

Then, by (2.74) and (2.75), the desired equality (2.73) is equivalent to $x'_3 = x_3$ and $y'_3 = y_3$. Assume that the latter are false. Then, by (2.75) and (2.76),

$$x'_3 \text{ is the reflection of } x_3 \text{ in the straight line } x_2 y_2, \quad (2.77)$$

$$y'_3 \text{ is the reflection of } y_3 \text{ in the straight line } x_2y_2. \quad (2.78)$$

Suppose first that

$$]x_1, y_1[\cap]x_2, y_2[=]x_2, y_2[\cap]x_3, y_3[= \emptyset. \quad (2.79)$$

We can assume without loss of generality that x_1 and z_1^+ lie on one side of the straight line x_2x_3 , x_1 and x_2 lie on one side of the straight line $z_1^+z_2^-$, and x_2 and x_3 lie on one side of the straight line $z_2^+z_3^-$. It follows from Remark 2.4, (2.77), and (2.78) that

$$w' = w \text{ for all } w \in \Gamma_x^U(x_1, y_1, x_2, y_2, z_1^+, z_2^-) \cup \Gamma_y^U(x_1, y_1, x_2, y_2, z_1^+, z_2^-), \quad (2.80)$$

w' is the reflection of w in the straight line x_2y_2 for all

$$w \in \Gamma_x^U(x_3, y_3, x_2, y_2, z_3^-, z_2^+) \cup \Gamma_y^U(x_3, y_3, x_2, y_2, z_3^-, z_2^+). \quad (2.81)$$

Interchanging the domains U and V , if necessary, we can assume without loss of generality that x_1 , z_1^+ , and z_3^- lie on one side of the straight line x_2y_2 . It is geometrically obvious now that in our situation we always deal with at least one of the following three possibilities:

- (i) there is a point $u_0 \in \Gamma_x^U(x_1, y_1, x_2, y_2, z_1^+, z_2^-) \cap \Gamma_x^U(x_3, y_3, x_2, y_2, z_3^-, z_2^+) \setminus \{x_2\}$;
- (ii) there is a point $u_0 \in \Gamma_x^U(x_1, y_1, x_2, y_2, z_1^+, z_2^-)$ such that $]x_2, u_0[$ is a boundary interval intersecting $]z_2^+, z_3^-[\cup]x_3, y_3[$ and, thereby, the interval $]x_2, u_0[$ is U -joinable with the interval $]x_3, y_3[$;
- (iii) there is a point $u_0 \in \Gamma_x^U(x_3, y_3, x_2, y_2, z_3^-, z_2^+)$ such that $]x_2, u_0[$ is a boundary interval intersecting $]z_1^+, z_2^-[\cup]x_1, y_1[$ and, thereby, the interval $]x_2, u_0[$ is U -joinable with the interval $]x_1, y_1[$.

But, by Lemma 2.13 and Remark 2.4, each of these assertions implies the following:

- (iv) there is a point $u_0 \in \Gamma_x^U(x_1, y_1, x_2, y_2, z_1^+, z_2^-) \cup \Gamma_x^U(x_3, y_3, x_2, y_2, z_3^-, z_2^+)$ such that u_0 does not lie on the straight line x_2y_2 and $|u'_0 - x'_i| = |u_0 - x_i|$ and $|u'_0 - y'_i| = |u_0 - y_i|$ for all $i = 1, 2, 3$.

However, this obviously contradicts (2.77), (2.78), (2.80), and (2.81). This contradiction completes the proof of Lemma 2.14 in the case when (2.79) are valid. If equalities (2.79) are false then the proof is carried out in a similar way; moreover, it is even simpler: we only need to use Lemma 2.12 and Remark 2.2. We skip the corresponding computation in view of their simplicity.

Definition 2.1. We say that boundary intervals $]x, y[$ and $]w, z[$ (with respect to U) are U -equivalent if there is a sequence of boundary intervals

$]x_i, y_i[$, $i = 1, \dots, m$, such that $x_1 = x$, $y_1 = y$, $x_m = w$, $y_m = z$, and the intervals $]x_i, y_i[$ and $]x_{i+1}, y_{i+1}[$ are U -joinable for all $i = 1, \dots, m-1$.

Now, we can restate Lemma 2.14 as follows:

Lemma 2.15 *Boundary intervals $]x, y[$ and $]w, z[$ (with respect to U) are U -equivalent if and only if the intervals $]x', y'[$ and $]w', z'[$ are V -equivalent. If boundary intervals $]x, y[$ and $]w, z[$ are U -equivalent then the quadrangle $xywz$ is Euclideanly isometric to the quadrangle $x'y'w'z'$.*

Lemma 2.16 *Boundary intervals $]x, y[$ and $]w, z[$ (with respect to U) are U -equivalent if and only if there is a component U_i (see the definition of U_i in Section 1) such that $]x, y[\cup]w, z[\subset \text{cl } U_i$.*

Proof is simple and carried out by elementary means; therefore, we omit it.

Lemma 2.17 *Let $\tilde{x} \in (\partial_H U_i) \cap (\partial_H U)$ and $x = p\tilde{x}$. Then there are sequences $\{x_\nu\}_{\nu \in \mathbb{N}}$, $\{y_\nu\}_{\nu \in \mathbb{N}}$ of points such that $]x_\nu, y_\nu[$ are boundary intervals for U and $]x_\nu, y_\nu[\subset \text{cl } U_i$; moreover, $\rho_{\partial_H U, U}(\tilde{x}_\nu, \tilde{x}) \leq \rho_{\partial_H U_i, U_i}(\tilde{x}_\nu, \tilde{x}) \rightarrow 0$ as $\nu \rightarrow \infty$.*

In connection with Lemma 2.17, observe that by $(\partial_H U_i) \cap (\partial_H U)$ we naturally mean the set of elements $\tilde{x} \in \partial_H U$ such that there is a sequence of points $w_\nu \in U_i$ which is Cauchy with respect to the intrinsic metric of U_i and such that $\rho_{U_H}(w_\nu, \tilde{x}) \rightarrow 0$.

Proof of Lemma 2.17. Take a sequence w_ν . Then from the corresponding definitions we immediately see that $\rho_{U_H}(w_\nu, \tilde{x}) \leq \rho_{U_{i_H}}(w_\nu, \tilde{x}) \rightarrow 0$ as $\nu \rightarrow \infty$.

Put $R_\nu = \text{dist}(w_\nu, \partial U)$. By the choice of w_ν and R_ν , we find that $R_\nu \rightarrow 0$ as $\nu \rightarrow \infty$ and

$$\forall \nu \quad (B(w_\nu, R_\nu) \subset U \ \& \ \exists x_\nu \in \text{cl } B(w_\nu, R_\nu) \cap \partial U). \quad (2.82)$$

By construction, $]x_\nu, w_\nu[\subset U_i$, $x_\nu \in \partial U_i$. If we take a sequence of points of the (open) interval $]x_\nu, w_\nu[$ converging to x_ν then it generates some element of the Hausdorff boundary $\tilde{x}_\nu \in (\partial_H U_i) \cap (\partial_H U)$. Hence, we find in turn that $\rho_{U_H}(\tilde{x}_\nu, w_\nu) = \rho_{U_{i_H}}(\tilde{x}_\nu, w_\nu) = R_\nu \rightarrow 0$ as $\nu \rightarrow \infty$. It is geometrically obvious now that

$$\forall \nu \quad \exists y_\nu \in \partial U \quad (]x_\nu, y_\nu[\subset U \ \& \]x_\nu, y_\nu[\cap B(w_\nu, R_\nu) \neq \emptyset). \quad (2.83)$$

Indeed, if this were false then, by (2.82), the half-plane passing through the point x_ν and containing the ball $B(w_\nu, R_\nu)$ would lie entirely in U . But this contradicts the membership $w_\nu \in U_i$ and the definition of U_i .

Thus, (2.83) is valid. Then from the definition of U_i and the previous constructions we immediately find that $]x_\nu, y_\nu[\subset \text{cl } U_i$. The last inclusion together with the facts established above yields the claim of Lemma 2.17.

Lemma 2.18 *We can enumerate the components U_i and V_i so that this enumeration determines a bijective correspondence between the components U_i and V_i ; moreover, there exist Euclideanly isometric mappings $Q_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $Q_i(U_i) = V_i$ for each component U_i . Now, $\tilde{x} \in (\partial_H U_i) \cap (\partial_H U)$ if and only if $f(\tilde{x}) \in (\partial_H V_i) \cap (\partial_H V)$; moreover, once these containments hold, we also have $x' = Q_i(x)$, where $x = p\tilde{x}$. Finally, $y \in U \cap \partial U_i$ if and only if $Q_i(y) \in V \cap \partial V_i$.*

Lemma 2.18 is a simple consequence of Lemmas 2.1 and 2.15-2.17; details are omitted. Henceforth we assume that the components U_i and V_i are enumerated as in Lemma 2.18.

From Lemma 2.18 we immediately obtain the following:

Lemma 2.19 *If $F_U = \emptyset$ (see the definition of F_U in Section 1) then U is determined uniquely from the relative metric of its Hausdorff boundary.*

Remark 2.5. Now, by Lemma 2.19, we suppose unless the contrary is specified that $F_U \neq \emptyset$.

Remark 2.6. Let $x \in (\partial F_U) \cap (\partial U)$. Then, obviously, there is a unique⁶ element $\tilde{x} \in (\partial_H F_U) \cap (\partial_H U)$ such that $p\tilde{x} = x$. In this case we put $x' = pf(\tilde{x})$ unless the contrary is specified.

From Lemma 2.18 and Remark 2.6 we also easily obtain the following:

Lemma 2.20 *If $x \in (\partial F_U) \cap (\partial U)$ then $f(\tilde{x}) \in (\partial_H F_V) \cap (\partial_H V)$ and, in particular, $x' \in (\partial F_U) \cap (\partial U)$. If $x \in (\partial F_U) \setminus (\partial U)$ then there is a unique number $i = i(x)$ such that $x \in U \cap \partial U_i$ and $Q_i(x) \in V \cap \partial V_i$.*

The following geometric lemma is trivial:

Lemma 2.21 *The boundary ∂F_U has at most two connected components. Moreover, if the boundary ∂U has two connected components then these connected components are two parallel straight lines and the set F_U itself represents the union of two half-planes by these straight lines.*

⁶Here we use Remark 2.3.

Lemma 2.21 will help us to derive the two lemmas:

Lemma 2.22 *The boundary ∂F_U contains two connected components if and only if the boundary ∂F_V contains two connected components.*

Proof bases on using Lemmas 2.18, 2.20, and 2.21 together with the general facts on the continuity of f . We skip the details of computations in view of their simplicity.

Lemma 2.23 *If the boundary ∂F_U is disconnected then U is determined uniquely from the relative metric of its Hausdorff boundary.*

Proof is carried out by simple application of Lemmas 2.18 and 2.20-2.22.

Lemma 2.24 *Suppose that each of the boundaries ∂F_U and ∂F_V is a connected set (i.e., a convex curve). Define the mapping $\theta : \partial F_U \rightarrow \mathbb{R}^2$ as*

$$\theta(w) = \begin{cases} w', & w \in (\partial F_U) \cap (\partial U); \\ Q_i(w), & w \in U \cap \partial U_i. \end{cases}$$

Then θ is a homeomorphism between the convex curves ∂F_U and ∂F_V ; moreover, this homeomorphism preserves the arc length (i.e., for arbitrary two points x and y the length of the arc connecting them in ∂F_U coincides with the length of the image of this arc under the mapping θ).

Proof. The fact that θ is a homeomorphism of the convex curves ∂F_U and ∂F_V follows from construction and Lemmas 2.18 and 2.20 (also see Remarks 2.6 and 2.1). Now, prove that this homeomorphism preserves arc length. Take an arbitrary pair of points $x, y \in \partial F_U$. Since all Q_i are isometries, by the construction of θ , we can assume without loss of generality that $x, y \in \partial U$. Denote by Γ the convex arc joining x and y in ∂F_U and put $\Gamma_1 = \Gamma \cap \partial U$. Let $\gamma : [0, \Phi] \rightarrow \Gamma$ be some parametrization of the arc Γ .

Since Γ is a convex arc whose all extreme points lie in Γ_1 , it is geometrically obvious that, for each $\varepsilon > 0$, there is a partition $0 < \varphi_0 < \varphi_1 < \dots < \varphi_N = \Phi$, $N = N(\varepsilon)$, such that $\forall \nu = 0, 1, \dots, N$ $\gamma(\varphi_\nu) \in \Gamma_1$ and

$$\forall \nu = 0, 1, \dots, N-1 \quad \frac{l(\gamma([\varphi_\nu, \varphi_{\nu+1}]))}{|\gamma(\varphi_\nu) - \gamma(\varphi_{\nu+1})|} \leq 1 + \varepsilon. \quad (2.84)$$

Using the definition of Hausdorff metric and the definition of F_U , we find that

$$\forall \nu = 0, 1, \dots, N-1 \quad \rho_{V_H}(f(\widetilde{\gamma(\varphi_\nu)}), f(\widetilde{\gamma(\varphi_{\nu+1})})) \leq l(\theta(\gamma([\varphi_\nu, \varphi_{\nu+1}]))). \quad (2.85)$$

From (2.84) and (2.85) we conclude that

$$\begin{aligned} \forall \nu = 0, 1, \dots, N-1 \quad & \frac{1}{1+\varepsilon} l(\gamma([\varphi_\nu, \varphi_{\nu+1}])) \leq |\gamma(\varphi_\nu) - \gamma(\varphi_{\nu+1})| \leq \\ \rho_{U_H}(\widetilde{\gamma(\varphi_\nu)}, \widetilde{\gamma(\varphi_{\nu+1})}) &= \rho_{V_H}(f(\widetilde{\gamma(\varphi_\nu)}), f(\widetilde{\gamma(\varphi_{\nu+1})})) \leq l(\theta(\gamma([\varphi_\nu, \varphi_{\nu+1}]))). \end{aligned} \quad (2.86)$$

It follows from the arbitrariness of $\varepsilon > 0$ and (2.86) that $l(\Gamma) \leq l(\theta(\Gamma))$. The reverse inequality follows, if we interchange the domains U and V . Lemma 2.24 is proven.

Proof of Item (II) of Theorem 1.1. Necessity in Item (II) follows from Item (I) proven above and Lemmas 2.3, 2.18, 2.23, and 2.24. Sufficiency in Item (II) is obvious: in the presence of the homeomorphism $\theta : \partial F_U \rightarrow \partial F_V$ and the isometries $Q_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with properties (IIa)-(IIc) we can naturally construct the isometry $f : \partial_H U \rightarrow \partial_H V$.

Proof of Theorem 1.2. Let U be a convex domain in \mathbb{R}^n , $n \geq 2$, different from a half-space. Under these conditions, we can identify the elements of the Hausdorff boundary $\partial_H U$ with the elements of the usual Euclidean boundary ∂U . Suppose that the domain $V \subset \mathbb{R}^n$ is such that there is an isometry $f : \partial_H U \rightarrow \partial_H V$. Then the isometry f naturally generates the mapping $g : \partial U \rightarrow \partial V$ by the formula $g(x) = pf(x)$. It is easily seen that, by convexity of U , the resulting mapping g is 1-Lipschitz. We have the following elementary lemma:

Lemma 2.25 *Let U be a convex domain in \mathbb{R}^n , $n \geq 2$, different from a half-space and take $[x, y] \subset U$. Then there is a sequence of boundary intervals $]x_\nu, y_\nu[$ (with respect to U) such that $\forall z \in [x, y] \quad \text{dist}(z, [x_\nu, y_\nu]) (= \inf_{w \in [x_\nu, y_\nu]} |z - w|) \rightarrow 0$ as $\nu \rightarrow \infty$.*

It follows from this lemma, Lemma 2.3, and the 1-Lipschitz continuity of g that, in fact, $|g(x) - g(y)| = |x - y|$ for all $x, y \in \partial U$. Hence, there is a Euclidean isometry $Q : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $Q(x) = g(x)$ for all $x \in \partial U$. It is easily seen that $Q(U) = V$. The proof of Theorem 1.2 is complete.

3 Rigidity conditions for the boundaries of submanifolds in a Riemannian manifold

RIGIDITY PROBLEMS AND INTRINSIC GEOMETRY OF SUBMANIFOLDS IN RIEMANNIAN MANIFOLDS

Let (X, g) be an n -dimensional smooth connected Riemannian manifold without boundary and let Y be its n -dimensional compact connected C^0 -submanifold with nonempty boundary ∂Y ($n \geq 2$).

A classical object of investigations (see, for example, [11]) is given by the intrinsic metric $\rho_{\partial Y}$ on a hypersurface ∂Y defined for $x, y \in \partial Y$ as the infimum of the lengths of curves $\nu \subset \partial Y$ joining x and y . In the recent decades, an alternative approach arose in the rigidity theory for submanifolds of Riemannian manifolds (see, for instance, the recent articles [9], [12], and [13], which also contain a historical survey of works on the topic). In accordance with this approach, the metric on ∂Y is induced by the intrinsic metric of the interior $\text{Int } Y$ of the submanifold Y .

Namely, suppose that Y satisfies the following condition:

(i) if $x, y \in Y$, then

$$\rho_Y(x, y) = \liminf_{x' \rightarrow x, y' \rightarrow y; x', y' \in \text{Int } Y} \{\inf[l(\gamma_{x', y', \text{Int } Y})]\} < \infty, \quad (3.1)$$

where $\inf[l(\gamma_{x', y', \text{Int } Y})]$ is the infimum of the lengths $l(\gamma_{x', y', \text{Int } Y})$ of smooth paths $\gamma_{x', y', \text{Int } Y} : [0, 1] \rightarrow \text{Int } Y$ joining x' and y' in the interior $\text{Int } Y$ of Y .

Remark 3.1. Easy examples show that if X is an n -dimensional connected smooth Riemannian manifold without boundary then an n -dimensional compact connected C^0 -submanifold in X with nonempty boundary may fail to satisfy condition (i). For $n = 2$, we have the following counterexample:

Let (X, g) be the space \mathbb{R}^2 equipped with Euclidean metric and let Y be a closed Jordan domain in \mathbb{R}^2 whose boundary is the union of the singleton $\{0\}$ consisting of the origin 0 , the segment $\{(1-t)(e_1 + 2e_2) + t(e_1 + e_2) : 0 \leq t \leq 1\}$, and of the segments of the following four types:

$$\begin{aligned} & \left\{ \frac{(1-t)(e_1 + e_2)}{n} + \frac{te_1}{n+1} : 0 \leq t \leq 1 \right\} \quad (n = 1, 2, \dots); \\ & \left\{ \frac{e_1 + (1-t)e_2}{n} : 0 \leq t \leq 1 \right\} \quad (n = 2, 3, \dots); \\ & \left\{ \frac{(1-t)(e_1 + 2e_2)}{n} + \frac{2t(2e_1 + e_2)}{4n+3} : 0 \leq t \leq 1 \right\} \quad (n = 1, 2, \dots); \\ & \left\{ \frac{(1-t)(e_1 + 2e_2)}{n+1} + \frac{2t(2e_1 + e_2)}{4n+3} : 0 \leq t \leq 1 \right\} \quad (n = 1, 2, \dots). \end{aligned}$$

Here e_1, e_2 is the canonical basis in \mathbb{R}^2 . By the construction of Y , we have $\rho_Y(0, E) = \infty$ for every $E \in Y \setminus \{0\}$ (see figure 1).

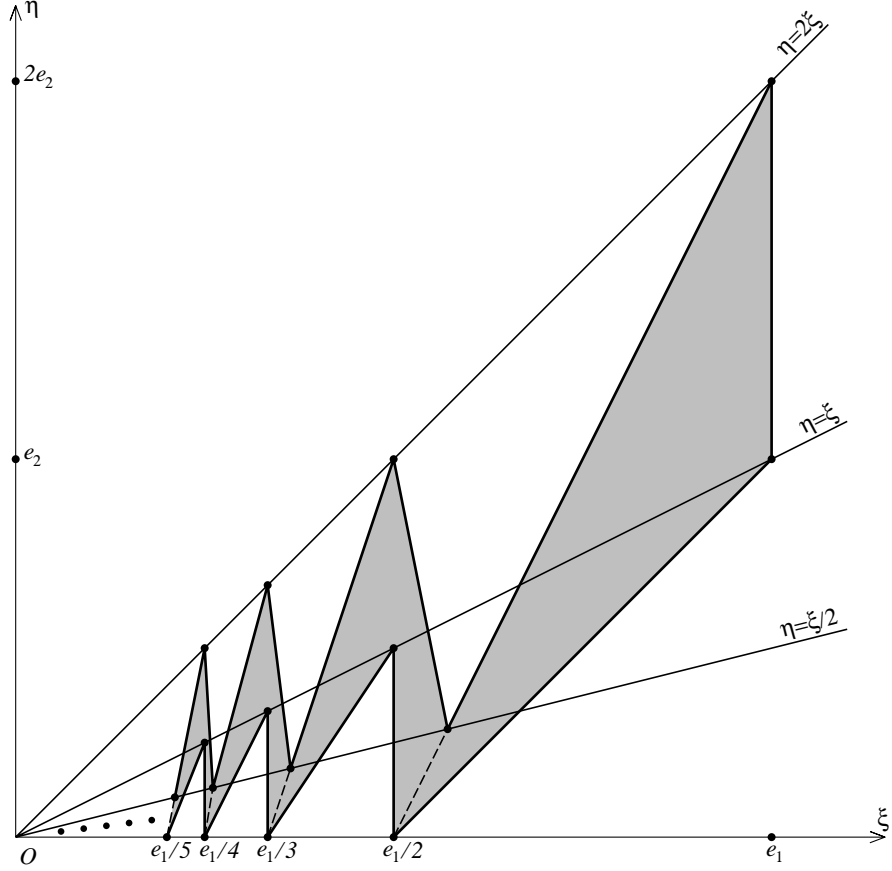


Figure 1: An example of 2-dimensional compact connected C^0 -submanifold with nonempty boundary which does not satisfy condition (i)

Remark 3.2. Note that if $X = \mathbb{R}^n$ and U is a domain in \mathbb{R}^n whose closure $Y = \text{cl } U$ is a Lipschitz manifold (such that $\partial(\text{cl } U) = \partial U \neq \emptyset$), then $\rho_{\partial U, U}(x, y) = \rho_Y(x, y)$ ($x, y \in \partial U$) and Y satisfies (i). Hence, this example is an important particular case of submanifolds Y in a Riemannian manifold X satisfying (i).

To prove our rigidity results for boundaries of submanifolds in a Riemannian manifold, we first need to study the properties of the intrinsic geometry of these submanifolds.

One of the main results of this section is as follows:

Theorem 3.1 *Let $n = 2$. Then, under condition (i), the function ρ_Y defined by 3.1 is a metric on Y .*

Proof. It suffices to prove that ρ_Y satisfies the triangle inequality. Let A , O , and D be three points on the boundary of Y (note that this case is basic because the other cases are simpler). Consider $\varepsilon > 0$ and assume that $\gamma_{A_\varepsilon O_\varepsilon^1} : [0, 1] \rightarrow \text{Int } Y$ and $\gamma_{O_\varepsilon^2 D_\varepsilon} : [2, 3] \rightarrow \text{Int } Y$ are smooth paths with the endpoints $A_\varepsilon = \gamma_{A_\varepsilon O_\varepsilon^1}(0)$, $O_\varepsilon^1 = \gamma_{A_\varepsilon O_\varepsilon^1}(1)$ and $D_\varepsilon = \gamma_{O_\varepsilon^2 D_\varepsilon}(3)$, $O_\varepsilon^2 = \gamma_{O_\varepsilon^2 D_\varepsilon}(2)$ satisfying the conditions $\rho_X(A_\varepsilon, A) \leq \varepsilon$, $\rho_X(D_\varepsilon, D) \leq \varepsilon$, $\rho_X(O_\varepsilon^j, O) \leq \varepsilon$ ($j = 1, 2$), $|l(\gamma_{A_\varepsilon O_\varepsilon^1}) - \rho_Y(A, O)| \leq \varepsilon$, and $|l(\gamma_{O_\varepsilon^2 D_\varepsilon}) - \rho_Y(O, D)| \leq \varepsilon$. Let (U, h) be a chart of the manifold X such that U is an open neighborhood of the point O in X , $h(U)$ is the unit disk $B(0, 1) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$ in \mathbb{R}^2 , and $h(O) = 0$ ($0 = (0, 0)$ is the origin in \mathbb{R}^2); moreover, $h : U \rightarrow h(U)$ is a diffeomorphism having the following property: there exists a chart (Z, ψ) of Y with $\psi(O) = 0$, $A, D \in U \setminus \text{cl}_X Z$ ($\text{cl}_X Z$ is the closure of Z in the space (X, g)) and $Z = \tilde{U} \cap Y$ is the intersection of an open neighborhood $\tilde{U} (\subset U)$ of O in X and Y whose image $\psi(Z)$ under ψ is the half-disk $B_+(0, 1) = \{(x_1, x_2) \in B(0, 1) : x_1 \geq 0\}$. Suppose that σ_r is an arc of the circle $\partial B(0, r)$ which is a connected component of the set $V \cap \partial B(0, r)$, where $V = h(Z)$ and $0 < r < r^* = \min\{|h(\psi^{-1}(x_1, x_2))| : x_1^2 + x_2^2 = 1/4, x_1 \geq 0\}$. Among these components, there is at least one (preserve the notation σ_r for it) whose ends belong to the sets $h(\psi^{-1}(\{-te_2 : 0 < t < 1\}))$ and $h(\psi^{-1}(\{te_2 : 0 < t < 1\}))$ respectively. Otherwise, the closure of the connected component of the set $V \cap B(0, r)$ whose boundary contains the origin would contain a point belonging to the arc $\{e^{i\theta}/2 : |\theta| \leq \pi/2\}$ (here we make use of the complex notation $z = re^{i\theta}$ for points $z \in \mathbb{R}^2 (= \mathbb{C})$). But this is impossible. Therefore, the above-mentioned arc σ_r exists.

It is easy to check that if ε is sufficiently small then the images of the paths $h \circ \gamma_{A_\varepsilon O_\varepsilon^1}$ and $h \circ \gamma_{O_\varepsilon^2 D_\varepsilon}$ also intersect the arc σ_r , i.e., there are $t_1 \in]0, 1[$, $t_2 \in]2, 3[$ such that $\gamma_{A_\varepsilon O_\varepsilon^1}(t_1) = x^1 \in Z$, $\gamma_{O_\varepsilon^2 D_\varepsilon}(t_2) = x^2 \in Z$ and $h(x^j) \in \sigma_r$, $j = 1, 2$. Let $\gamma_r : [t_1, t_2] \rightarrow \sigma_r$ be a smooth parametrization of the corresponding subarc of σ_r , i.e., $\gamma_r(t_j) = h(x^j)$, $j = 1, 2$. Now we can define a mapping $\tilde{\gamma}_\varepsilon : [0, 3] \rightarrow \text{Int } Y$ by setting

$$\tilde{\gamma}_\varepsilon(t) = \begin{cases} \gamma_{A_\varepsilon O_\varepsilon^1}(t), & t \in [0, t_1]; \\ h^{-1}(\gamma_r(t)), & t \in]t_1, t_2[; \\ \gamma_{O_\varepsilon^2 D_\varepsilon}(t), & t \in [t_2, 3]. \end{cases}$$

By construction, $\tilde{\gamma}_\varepsilon$ is a piecewise smooth path joining the points $A_\varepsilon = \tilde{\gamma}_\varepsilon(0)$, $D_\varepsilon = \tilde{\gamma}_\varepsilon(3)$ in $\text{Int } Y$; moreover,

$$l(\tilde{\gamma}_\varepsilon) \leq l(\gamma_{A_\varepsilon O_\varepsilon^1}) + l(\gamma_{O_\varepsilon^2 D_\varepsilon}) + l(h^{-1}(\sigma_r)).$$

By an appropriate choice of $\varepsilon > 0$, we can make $r > 0$ arbitrarily small, and since a piecewise smooth path can be approximated by smooth paths, we have $\rho_Y(A, D) \leq \rho_Y(A, O) + \rho_Y(O, D)$.

In connection with Theorem 3.1, there appears a natural question: Are there analogs of this theorem for $n \geq 3$? The following Theorem 3.2 answers this question in the negative:

Theorem 3.2 *If $n \geq 3$ then there exists an n -dimensional compact connected C^0 -manifold $Y \subset \mathbb{R}^n$ with nonempty boundary ∂Y such that the condition (i) (where now $X = \mathbb{R}^n$) is fulfilled for Y but the function ρ_Y in this condition is not a metric on Y .*

Proof. It suffices to consider the case of $n = 3$. Suppose that A, O, D are points in \mathbb{R}^3 , O is the origin in \mathbb{R}^3 , $|A| = |D| = 1$, and the angle between the segments OA and OD is equal to $\frac{\pi}{6}$.

The manifold Y will be constructed so that $O \in \partial Y$, and $]O, A[\subset \text{Int } Y$, $]O, D[\subset \text{Int } Y$. Under these conditions, $\rho_Y(O, A) = \rho_Y(O, D) = 1$. However, the boundary of Y will create "obstacles" between A and D such that the length of any curve joining A and D in $\text{Int } Y$ will be greater than $\frac{12}{5}$ (this means the violation of the triangle inequality for ρ_Y).

Consider a countable collection of mutually disjoint segments $\{I_j^k\}_{j \in \mathbb{N}, k=1, \dots, k_j}$ lying in the interior of the triangle $6\Delta AOD$ (which is obtained from the original triangle ΔAOD by dilation with coefficient 6) with the following properties:

(*) every segment $I_j^k = [x_j^k, y_j^k]$ lies on a ray starting at the origin, $y_j^k = 11x_j^k$, and $|x_j^k| = 2^{-j}$;

(**) any curve γ with ends A, D whose interior points lie in the interior of the triangle $4\Delta AOD$ and belong to no segment I_j^k , satisfies the estimate $l(\gamma) \geq 6$.

The existence of such a family of segments is certain: the segments of the family must be situated chequer-wise so that any curve disjoint from them be sawtooth, with the total length of its "teeth" greater than 6 (it can clearly be made greater than any prescribed positive number). However, below we exactly describe the construction.

It is easy to include the above-indicated family of segments in the boundary ∂Y of Y . Thus, it creates a desired "obstacle" to joining A and D in the plane of ΔAOD . But it makes no obstacle to joining A and D in the space. The simplest way to create such a space obstacle is as follows: Rotate each segment I_j^k along a spiral around the axis OA . Make the number of coils so large that the length of this spiral be large its pitch (i.e., the distance

between the origin and the the end of a coil) be sufficiently small. Then the set S_j^k obtained as the result of the rotation of the segment I_j^k is diffeomorphic to a plane rectangle, and it lies in a small neighborhood of the cone of revolution with axis AO containing the segment I_j^k . The last circumstance guarantees that the sets S_j^k are disjoint as before, and so (as above) it is easy to include them in the boundary ∂Y but, due to the properties of the I_j^k 's and a large number of coils of the spirals S_j^k , any curve joining A , D and disjoint from each S_j^k has length $\geq \frac{12}{5}$.

We turn to an exact description of the constructions used. First describe the construction of the family of segments I_j^k . They are chosen on the basis of the following observation:

Let $\gamma : [0, 1] \rightarrow 4\Delta AOD$ be any curve with ends $\gamma(0) = A$, $\gamma(1) = D$ whose interior points lie in the interior of the triangle $4\Delta AOD$. For $j \in \mathbb{N}$, put $R_j = \{x \in 4\Delta AOD : |x| \in [8 \cdot 2^{-j}, 4 \cdot 3^{-j}]\}$. It is clear that

$$4\Delta AOD \setminus \{O\} = \cup_{j \in \mathbb{N}} R_j.$$

Introduce the polar system of coordinates on the plane of the triangle ΔAOD with center O such that the coordinates of the points A, D are $r = 1, \varphi = 0$ and $r = 1, \varphi = \frac{\pi}{6}$, respectively. Given a point $x \in 6\Delta AOD$, let φ_x be the angular coordinate of x in $[0, \frac{\pi}{6}]$. Let $\Phi_j = \{\varphi_{\gamma(t)} : \gamma(t) \in R_j\}$. Obviously, there is $j_0 \in \mathbb{N}$ such that

$$\mathcal{H}^1(\Phi_{j_0}) \geq 2^{-j_0} \frac{\pi}{6}, \quad (3.2)$$

where \mathcal{H}^1 is the Hausdorff 1-measure. This means that, while in the layer R_{j_0} , the curve γ covers the angular distance $\geq 2^{-j_0} \frac{\pi}{6}$. The segments I_j^k must be chosen such that (3.2) together with the condition

$$\gamma(t) \cap I_j^k = \emptyset \quad \forall t \in [0, 1], \forall j \in \mathbb{N}, \forall k \in \{1, \dots, k_j\}$$

give the desired estimate $l(\gamma) \geq 6$. To this end, it suffices to take $k_j = [(2\pi)^j]$ (the integral part of $(2\pi)^j$) and

$$I_j^k = \{x \in 6\Delta AOD : \varphi_x = k(2\pi)^{-j} \frac{\pi}{6}, |x| \in [11 \cdot 2^{-j}, 2^{-j}]\},$$

$k = 1, \dots, k_j$. Indeed, under this choice of the I_j^k 's, estimate (3.2) implies that γ must intersect at least $(2\pi)^{j_0} 2^{-j_0} = \pi^{j_0} > 3^{j_0}$ of the figures

$$U_k = \{x \in R_{j_0} : \varphi_x \in]k(2\pi)^{-j_0} \frac{\pi}{6}, (k+1)(2\pi)^{-j_0} \frac{\pi}{6} [\}.$$

Since these figures are separated by the segments $I_{j_0}^k$ in the layer R_{j_0} , the curve γ must be disjoint from them each time in passing from one figure to another. The number of these passages must be at least $3^{j_0} - 1$, and a fragment of γ of length at least $2 \cdot 3 \cdot 2^{-j_0}$ is required for each passage (because the ends of the segments $I_{j_0}^k$ go beyond the boundary of the layer R_{j_0} containing the figures U_k at distance $3 \cdot 2^{-j_0}$). Thus, for all these passages, a section of γ is spent of length at least

$$6 \cdot 2^{-j_0} (3^{j_0} - 1) \geq 6.$$

Hence, the construction of the segments I_j^k satisfying (*) — (**) is finished.

Let us now describe the construction of the above-mentioned space spirals.

For $x \in \mathbb{R}^3$, denote by Π_x the plane that passes through x and is perpendicular to the segment OA . On $\Pi_{x_j^k}$, consider the polar coordinates (ρ, ψ) with origin at the point of intersection of $\Pi_{x_j^k}$ and $[O, A]$ (in this system, the point x_j^k has coordinates $\rho = \rho_j^k$, $\psi = 0$). Suppose that a point $x(\psi) \in \Pi_{x_j^k}$ moves along an Archimedean spiral, namely, the polar coordinates of the point $x(\psi)$ are $\rho(\psi) = \rho_j^k - \varepsilon_j \psi$, $\psi \in [0, 2\pi M_j]$, where ε_j is a small parameter to be specified below, and $M_j \in \mathbb{N}$ is chosen so large that the length of any curve passing between all coils of the spiral is at least 10.

Describe the choice of M_j more exactly. To this end, consider the points $x(2\pi)$, $x(2\pi(M_j - 1))$, $x(2\pi M_j)$, which are the ends of the first, penultimate, and last coils of the spiral respectively (with $x(0) = x_j^k$ taken as the starting point of the spiral). Then M_j is chosen so large that the following condition hold:

(*)₁ *The length of any curve on the plane $\Pi_{x_j^k}$, joining the segments $[x_j^k, x(2\pi)]$ and $[x(2\pi(M_j - 1)), x(2\pi M_j)]$ and disjoint from the spiral $\{x(\psi) : \psi \in [0, 2\pi M_j]\}$, is at least 10.*

Figuratively speaking, the constructed spiral bounds a "labyrinth", the mentioned segments are the entrance to and the exit from this labyrinth, and thus any path through the has length ≥ 10 .

Now, start rotating the entire segment I_j^k in space along the above-mentioned spiral, i.e., assume that $I_j^k(\psi) = \{y = \lambda x(\psi) : \lambda \in [1, 11]\}$. Thus, the segment $I_j^k(\psi)$ lies on the ray joining O with $x(\psi)$ and has the same length as the original segment $I_j^k = I_j^k(0)$. Define the surface $S_j^k = \bigcup_{\psi \in [0, 2\pi M_j]} I_j^k(\psi)$. This surface is diffeomorphic to a plane rectangle (strip). Taking $\varepsilon_j > 0$ sufficiently small, we may assume without loss of generality that $2\pi M_j \varepsilon_j$ is substantially less than ρ_j^k ; moreover, that the surfaces S_j^k

are mutually disjoint (obviously, the smallness of ε_j does not affect property $(*_1)$ which in fact depends on M_j).

Denote by $y(\psi) = 11x(\psi)$ the second end of the segment $I_j^k(\psi)$. Consider the trapezium P_j^k with vertices y_j^k , x_j^k , $x(2\pi M_j)$, $y(2\pi M_j)$ and sides I_j^k , $I_j^k(2\pi M_j)$, $[x_j^k, x(2\pi M_j)]$, and $[y_j^k, y(2\pi M_j)]$ (the last two sides are parallel since they are perpendicular to the segment AO). By construction, P_j^k lies on the plane AOD ; moreover, taking ε_j sufficiently small, we can obtain the situation where the trapeziums P_j^k are mutually disjoint (since $P_j^k \rightarrow I_j^k$ under fixed M_j and $\varepsilon_j \rightarrow 0$). Take an arbitrary triangle whose vertices lie on P_j^k and such that one of these vertices is also a vertex at an acute angle in P_j^k . By construction, this acute angle is at least $\frac{\pi}{2} - \angle AOD = \frac{\pi}{3}$. Therefore, the ratio of the side of the triangle lying inside the trapezium P_j^k to the sum of the other two sides (lying on the corresponding sides of P_j^k) is at least $\frac{1}{2} \sin \frac{\pi}{3} > \frac{2}{5}$. If we consider the same ratio for the case of a triangle with a vertex at an obtuse angle of P_j^k then it is greater than $\frac{1}{2}$. Thus we have the following property:

*(*_2) For arbitrary triangle whose vertices lie on the trapezium P_j^k and one of these vertices is also a vertex in P_j^k , the sum of length of the sides situated on the corresponding sides of P_j^k is less than $\frac{5}{2}$ of the length of the third side (lying inside P_j^k).*

Let a point x lie inside the cone K formed by the rotation of the angle $\angle AOD$ around the ray OA . Denote by $\text{Proj}_{\text{rot}} x$ the point of the angle $\angle AOD$ which is the image of x under this rotation. Finally, let $K_{4\Delta AOD}$ stand for the corresponding truncated cone obtained by the rotation of the triangle $4\Delta AOD$, i.e., $K_{4\Delta AOD} = \{x \in K : \text{Proj}_{\text{rot}} x \in 4\Delta AOD\}$.

The key ingredient in the proof of our theorem is the following assertion:

*(*_3) For arbitrary space curve γ of length less than 10 joining the points A and D , contained in the truncated cone $K_{4\Delta AOD} \setminus \{O\}$, and disjoint from each strip S_j^k , there exists a plane curve $\tilde{\gamma}$ contained in the triangle $4\Delta AOD \setminus \{O\}$, that joins A and D , is disjoint from all segments I_j^k and such that the length of $\tilde{\gamma}$ is less than $\frac{5}{2}$ of the length of $\text{Proj}_{\text{rot}} \gamma$.*

Prove $(*_3)$. Suppose that its hypotheses are fulfilled. In particular, assume that the inclusion $\text{Proj}_{\text{rot}} \gamma \subset 4\Delta AOD \setminus \{O\}$ holds. We need to modify $\text{Proj}_{\text{rot}} \gamma$ so that the new curve be contained in the same set but be disjoint from each of the I_j^k 's. The construction splits into several steps.

Step 1. If $\text{Proj}_{\text{rot}} \gamma$ intersects a segment I_j^k then it necessarily intersects also at least one of the shorter sides of P_j^k .

Recall that, by construction, $P_j^k = \text{Proj}_{\text{rot}} S_j^k$; moreover, γ intersects no

spiral strip S_j^k . If $\text{Proj}_{\text{rot}} \gamma$ intersected P_j^k without intersecting its shorter sides then γ would pass through all coils of the corresponding spiral. Then, by $(*)_1$, the length of the corresponding fragment of γ would be ≥ 10 in contradiction to our assumptions. Thus, the assertion of step 1 is proved.

Step 2. Denote by $\gamma_{P_j^k}$ the fragment of the plane curve $\text{Proj}_{\text{rot}} \gamma$ beginning at the first point of its entrance into the trapezium P_j^k to the point of its exit from P_j^k (i.e., to its last intersection point with P_j^k). Then this fragment $\gamma_{P_j^k}$ can be deformed changing the first and the last points so that the corresponding fragment of the new curve lie entirely on the union of the sides of P_j^k ; moreover, its length is less than $\frac{5}{2}$ of the length of $\gamma_{P_j^k}$.

The assertion of step 2 immediately follows from the assertions of step 1 and $(*)_2$.

The assertion of step 2 in turn directly implies the desired assertion $(*)_3$. The proof of $(*)_3$ is finished.

Now, we are ready to pass to the final part of the proof of Theorem 3.2.

$(*)_4$ *The length of any space curve $\gamma \subset \mathbb{R}^3 \setminus \{O\}$ joining A and D and disjoint from each strip S_j^k is at least $\frac{12}{5}$.*

Prove the last assertion. Without loss of generality, we may also assume that all interior points of γ are inside the cone K (otherwise the initial curve can be modified without any increase of its length so that assumptions of $(*)_4$ are still fulfilled and the modified curve lies in K). If γ is not included in the truncated cone $K_{4\Delta AOD} \setminus \{O\}$ then $\text{Proj}_{\text{rot}} \gamma$ intersects the segment $[4A, 4D]$; consequently, the length of γ is at least $2(4 \sin \angle OAD - 1) = 2(4 \sin \frac{\pi}{3} - 1) = 2(2\sqrt{3} - 1) > 4$, and the desired estimate is fulfilled. Similarly, if the length of γ is at least 10 then the desired estimate is fulfilled automatically, and there is nothing to prove. Hence, we may further assume without loss of generality that γ is included in the truncated cone $K_{4\Delta AOD} \setminus \{O\}$ and its length is less than 10. Then, by $(*)_3$, there is a plane curve $\tilde{\gamma}$ contained in the triangle $4\Delta AOD \setminus \{O\}$, joining the points A and D , disjoint from each segment I_j^k , and such that the length of $\tilde{\gamma}$ is at most $\frac{5}{2}$ of the length of $\text{Proj}_{\text{rot}} \gamma$. By property $(**)$ of the family of segments I_j^k , the length of $\tilde{\gamma}$ is at least 6. Consequently, the length of $\text{Proj}_{\text{rot}} \gamma$ is at least $\frac{12}{5}$, which implies the desired estimate. Assertion $(*)_4$ is proved.

The just-proven property $(*)_4$ of the constructed objects implies Theorem 3.2. Indeed, since the strips S_j^k are mutually disjoint and, outside every neighborhood of the origin O , there are only finitely many of these strips, it is easy to construct a C^0 -manifold $Y \subset \mathbb{R}^3$ that is homeomorphic to a closed ball (i.e., ∂Y is homeomorphic to a two-dimensional sphere) and has the following properties:

- (I) $O \in \partial Y$, $[A, O[\cup]D, O[\subset \text{Int } Y$;
- (II) for every point $y \in (\partial Y) \setminus \{O\}$, there exists a neighborhood $U(y)$ such that $U(y) \cap \partial Y$ is C^1 -diffeomorphic to the plane square $[0, 1]^2$;
- (III) $S_j^k \subset \partial Y$ for all $j \in \mathbb{N}$, $k = 1, \dots, k_j$.

The construction of Y with properties (I)—(III) can be carried out, for example, as follows: As the surface of the zeroth step, take a sphere containing O and such that A and D are inside the sphere. At the j th step, a small neighborhood of the point O of our surface is smoothly deformed so that the modified surface is still smooth, homeomorphic to a sphere, and contains all strips S_j^k , $k = 1, \dots, k_j$. Besides, we make sure that, at the each step, the so-obtained surface be disjoint from the half-intervals $[A, O[$ and $]D, O[$, and, as above, contain all strips S_j^k , $i \leq j$, already included therein. Since the neighborhood we are deforming contracts to the point O as $j \rightarrow \infty$, the so-constructed sequence of surfaces converges (for example, in the Hausdorff metric) to a limit surface which is the boundary of a C^0 -manifold Y with properties (I)—(III).

Property (I) guarantees that $\rho_Y(A, O) = \rho_Y(A, D) = 1$ and $\rho_Y(O, x) \leq 1 + \rho_Y(A, x)$ for all $x \in Y$. Property (II) implies the estimate $\rho_Y(x, y) < \infty$ for all $x, y \in Y \setminus \{O\}$, which, granted the previous estimate, yields $\rho_Y(x, y) < \infty$ for all $x, y \in Y$. However, property (III) and the assertion $(*_4)$ imply that $\rho_Y(A, D) \geq \frac{12}{5} > 2 = \rho_Y(A, O) + \rho_Y(A, D)$. Theorem 3.2 is proved.

If ρ_Y is a metric (the dimension n (≥ 2) is arbitrary) then the question of the existence of geodesics is solved in the following assertion, which implies that ρ_Y is an *intrinsic metric* (see, for example, §6 from [3]).

Theorem 3.3 *Assume that ρ_Y is a finite function and is a metric on Y . Then any two points $x, y \in Y$ can be joined in Y by a shortest curve $\gamma : [0, L] \rightarrow Y$ in the metric ρ_Y ; i.e., $\gamma(0) = x$, $\gamma(L) = y$, and*

$$\rho_Y(\gamma(s), \gamma(t)) = t - s, \quad \forall s, t \in [0, L], \quad s < t. \quad (3.3)$$

Proof. Fix a pair of distinct points $x, y \in Y$ and put $L = \rho_Y(x, y)$. Now, take a sequence of paths $\gamma_j : [0, L] \rightarrow Y$ such that $\gamma_j(0) = x_j$, $\gamma_j(L) = y_j$, $x_j \rightarrow x$, $y_j \rightarrow y$, and $l(\gamma_j) \rightarrow L$ as $j \rightarrow \infty$. Without loss of generality, we may also assume that the parameterizations of the curves γ_j are their natural parameterizations up to a factor (tending to 1) and the mappings γ_j converge uniformly to a mapping $\gamma : [0, L] \rightarrow Y$ with $\gamma(0) = x$, $\gamma(L) = y$. By these assumptions,

$$\lim_{j \rightarrow \infty} l(\gamma_j|_{[s, t]}) = t - s \quad \forall s, t \in [0, L], \quad s < t. \quad (3.4)$$

Take an arbitrary pair of numbers $s, t \in [0, L]$, $s < t$. By construction, we have the convergence $\gamma_j(s) \in \text{Int } Y \rightarrow \gamma(s)$, $\gamma_j(t) \in \text{Int } Y \rightarrow \gamma(t)$ as $j \rightarrow \infty$. From here and the definition of the metric $\rho_Y(\cdot, \cdot)$ it follows that $\rho_Y(\gamma(s), \gamma(t)) \leq \lim_{j \rightarrow \infty} l(\gamma_j|_{[s, t]})$. By 3.4,

$$\rho_Y(\gamma(s), \gamma(t)) \leq t - s \quad \forall s, t \in [0, L], \quad s < t. \quad (3.5)$$

Prove that (3.5) is indeed an equality. Assume that $\rho_Y(\gamma(s'), \gamma(t')) < t' - s'$ for some $s', t' \in [0, L]$, $s' < t'$. Then applying the triangle inequality and then (3.5), we infer

$$\rho_Y(x, y) \leq \rho_Y(x, \gamma(s')) + \rho_Y(\gamma(s'), \gamma(t')) + \rho_Y(\gamma(t'), y) < s' + (t' - s') + (L - t') = L,$$

which contradicts the initial equality $\rho_Y(x, y) = L$. The so-obtained contradiction completes the proof of identity (3.3). q.e.d.

Remark 3.3. Identity (3.3) means that the curve of Theorem 3.3 is a geodesic in the metric ρ_Y , i.e., the length of its fragment between points $\gamma(s)$, $\gamma(t)$ calculated in ρ_Y is equal to $\rho_Y(\gamma(s), \gamma(t)) = t - s$. Nevertheless, if we compute the length of the above-mentioned fragment of the curve in the initial Riemannian metric then this length need not coincide with $t - s$; only the easily verifiable estimate $l(\gamma|_{[s, t]}) \leq t - s$ holds (see (3.4)). In the general case, the equality $l(\gamma|_{[s, t]}) = t - s$ can only be guaranteed if $n = 2$ (if $n \geq 3$ then the corresponding counterexample is constructed by analogy with the counterexample in the proof of Theorem 3.2, see above). In particular, though, by Theorem 3.3, the metric ρ_Y is always intrinsic in the sense of the definitions in [3, §6], the space (Y, ρ_Y) may fail to be a *space with intrinsic metric* in the sense of [ibid].

RIGIDITY THEOREMS FOR THE BOUNDARIES OF SUBMANIFOLDS IN RIEMANNIAN MANIFOLDS

As in the 1st part of Sec. 3, let (X, g) be an n -dimensional smooth connected Riemannian manifold without boundary and let ρ_X be its intrinsic metric (i.e., let $\rho_X(x, y)$ be the infimum of the lengths $l(\gamma_{x, y, X})$ of smooth paths $\gamma_{x, y, X} : [0, 1] \rightarrow X$ joining points x and y in a manifold X).

Assume that Y is an n -dimensional compact connected C^0 -submanifold with nonempty boundary ∂Y in X satisfying condition (i) in the 1st part of Sec. 3, moreover, ρ_Y is a metric on Y . Then Y is called strictly convex in the metric ρ_Y if, for any $\alpha, \beta \in Y$, any shortest path $\gamma = \gamma_{\alpha, \beta, Y} : [0, 1] \rightarrow Y$ between α and β (in the metric ρ_Y) satisfies $\gamma([0, 1]) \subset \text{Int } Y$.

Theorem 3.4 *Let $n = 2$. Assume that condition (i) holds for a 2-dimensional compact connected C^0 -submanifold Y_1 with nonempty boundary ∂Y_1 of a 2-dimensional smooth connected Riemannian manifold X without boundary*

which is strictly convex in the metric ρ_{Y_1} . Suppose that $Y_2 \subset X$ is also a 2-dimensional compact connected C^0 -submanifold of X with $\partial Y_2 \neq \emptyset$ satisfying (i); moreover, ∂Y_1 and ∂Y_2 are isometric in the metrics ρ_{Y_j} , for $j = 1, 2$. Then, Y_2 is strictly convex with respect to the metric ρ_{Y_2} .

Proof. Suppose that, for points $x, y \in Y_2$, there exists a shortest path $\gamma_{x,y,Y_2} : [0, 1] \rightarrow Y_2$ in the metric ρ_{Y_2} joining x and y and such that $\{\gamma_{x,y,Y_2}([0, 1])\} \cap \partial Y_2 \neq \emptyset$, i.e., $x' = \gamma_{x,y,Y_2}(t') \in \{\gamma_{x,y,Y_2}([0, 1]) \cap \partial Y_2\}$ for a point $t' \in]0, 1[$. By Theorem 3.3 and the fact that Y_2 is a 2-dimensional compact connected C^0 -submanifold in X , for a sufficiently small neighborhood of x' in Y_2 , we can find points $x_0, y_0 \in \partial Y_2$ and a shortest path $\gamma_{x_0,y_0,Y_2} : [0, 1] \rightarrow Y_2$ between x_0 and y_0 in the same metric satisfying the condition $x' \in \{\gamma_{x_0,y_0,Y_2}([0, 1]) \cap \partial Y_2\}$. Further, we will suppose that $x = x_0$ and $y = y_0$.

Now, assume that $f : \partial Y_1 \rightarrow \partial Y_2$ is an isometry of ∂Y_1 and ∂Y_2 in the metrics ρ_{Y_1} and ρ_{Y_2} of the boundaries ∂Y_1 and ∂Y_2 of the submanifolds Y_1 and Y_2 of X . From Theorem 3.3, we have

$$\rho_{Y_2}(x, x') + \rho_{Y_2}(x', y) = l_1 + l_2 = l = \rho_{Y_2}(x, y).$$

Since f is an isometry,

$$\rho_{Y_1}(f^{-1}(x), f^{-1}(x')) + \rho_{Y_1}(f^{-1}(x'), f^{-1}(y)) = \rho_{Y_2}(x, x') + \rho_{Y_2}(x', y).$$

Next, consider shortest paths $\gamma_{f^{-1}(x), f^{-1}(x'), Y_1} : [0, 1/2] \rightarrow Y_1$ and $\gamma_{f^{-1}(x'), f^{-1}(y), Y_1} : [1/2, 1] \rightarrow Y_1$ in ρ_{Y_1} between (respectively) $f^{-1}(x)$ and $f^{-1}(x')$ and $f^{-1}(x')$ and $f^{-1}(y)$, and then construct a path $\gamma : [0, 1] \rightarrow Y_1$ by setting $\gamma(t) = \gamma_{f^{-1}(x), f^{-1}(x'), Y_1}(t)$ if $0 \leq t < 1/2$ and $= \gamma_{f^{-1}(x'), f^{-1}(y), Y_1}(t)$ for $1/2 \leq t \leq 1$. Let $l_{Y_1}(\delta)$ be the length of a path $\delta : [0, 1] \rightarrow Y_1$ in the metric ρ_{Y_1} . Since ρ_{Y_1} is a metric on Y_1 , it is not difficult to show that

$$l_{Y_1}(\gamma) \leq l_{Y_1}(\gamma_{f^{-1}(x), f^{-1}(x'), Y_1}) + l_{Y_1}(\gamma_{f^{-1}(x'), f^{-1}(y), Y_1}) = l_1 + l_2.$$

Hence γ is a shortest path in ρ_{Y_1} joining $f^{-1}(x)$ and $f^{-1}(y)$ in Y_1 . This contradicts the strict convexity of Y_1 . The theorem is proved.

Corollary 3.1 *Suppose that the conditions of Theorem 3.4 hold and the manifold X has the following property: $\rho_X(x, y) = \rho_Y(x, y)$ for any two points x and y from every 2-dimensional compact connected C^0 -submanifold $Y \subset X$ with $\partial Y \neq \emptyset$ satisfying condition (i) and strictly convex with respect to the metric ρ_Y . Then, ∂Y_1 and ∂Y_2 are isometric in the metric ρ_X on the ambient manifold X .*

Remark 3.4. The condition imposed on the manifold X in Corollary 3.1 can be reformulated as follows: in this manifold, every 2-dimensional compact connected C^0 -submanifold Y with boundary satisfying condition (i) and strictly convex with respect to its intrinsic metric ρ_Y is a convex set in the ambient space X with respect to the metric ρ_X (for the notion of a convex set in a metric space the reader is referred, for example, to [11]).

We have the following analog of Theorem 3.4 and Corollary 3.1 (combined together) for $n \geq 3$.

Theorem 3.5 *Let $n \geq 3$. Suppose that (X, g) is an n -dimensional smooth connected Riemannian manifold without boundary and Y_1 and Y_2 are n -dimensional compact connected C^0 -submanifolds with nonempty boundaries ∂Y_1 and ∂Y_2 satisfying conditions (i),*

(ii) ρ_{Y_j} is a metric on Y_j ($j = 1, 2$),

and

(iii) for any two points $a, b \in Y_j$, there exist points $c, d \in \partial Y_j$ which can be joined in Y_j by a shortest path $\gamma : [0, 1] \rightarrow Y_j$ in the metric ρ_{Y_j} so that $a, b \in \gamma([0, 1])$.

Furthermore, assume that Y_1 is strictly convex in the metric ρ_{Y_1} , X has the additional property that $\rho_X(x, y) = \rho_Y(x, y)$ for any two points x and y in every n -dimensional compact connected C^0 -submanifold $Y \subset X$ with $\partial Y \neq \emptyset$ satisfying conditions (i)-(iii) and strictly convex with respect to ρ_Y and the boundaries ∂Y_1 and ∂Y_2 of the submanifolds Y_1 and Y_2 are isometric with respect to the metrics ρ_{Y_j} , where $j = 1, 2$. Then, ∂Y_1 and ∂Y_2 are isometric with respect to ρ_X .

Remark 3.5. For a submanifold Y in X , (i) and (ii) can be considered as conditions of generalized regularity near its boundary.

Remark 3.6. Theorem 3.4, Corollary 3.1, and Theorem 3.5 are closely related to a theorem of A. D. Aleksandrov about the rigidity of the boundary ∂U of a strictly convex domain U in Euclidean n -space \mathbb{R}^n ($n \geq 2$) by the relative metric $\rho_{\partial U, U}$ on the boundary (A. D. Aleksandrov's theorem was first published (with his consent) by V. A. Aleksandrov in [3]). The following is an important particular case of this theorem.

Theorem 3.6 *Let U_1 be a strictly convex domain in \mathbb{R}^n (i.e., for any $\alpha, \beta \in \text{cl } U_1$ every shortest path $\gamma = \gamma_{\alpha, \beta, \text{cl } U_1} : [0, 1] \rightarrow \text{cl } U_1$ between α and β (in the metric $\rho_{\text{cl } U_1}$) satisfies $\gamma([0, 1]) \subset U_1$). Assume that $U_2 \subset \mathbb{R}^n$ is any domain whose closure is a Lipschitz manifold (such that $\partial(\text{cl } U_2) = \partial U_2 \neq \emptyset$); moreover, ∂U_1 and ∂U_2 are isometric (globally) in their relative metrics $\rho_{\partial U_1, U_1}$ and $\rho_{\partial U_2, U_2}$. Then ∂U_1 and ∂U_2 are isometric in Euclidean metric.*

We say that an n -dimensional compact (closed) connected C^0 -submanifold Y with boundary $\partial Y \neq \emptyset$ of an n -dimensional smooth connected (respectively, n -dimensional smooth complete connected) Riemannian manifold X without boundary has property (o) if $\gamma_{x,y,Y}([0,1]) \subset \text{Int } Y$ for any two points $x, y \in \partial Y$ and for every shortest path $\gamma_{x,y,Y} : [0,1] \rightarrow Y$ in the metric ρ_Y joining these points.

Theorem 3.7 *Let $n = 2$. Suppose that (i) holds for a 2-dimensional compact connected C^0 -submanifold Y_1 with boundary $\partial Y_1 \neq \emptyset$ in a 2-dimensional smooth connected Riemannian manifold X without boundary; moreover, Y_1 has property (o). Assume that $Y_2 \subset X$ is a 2-dimensional compact connected C^0 -submanifold with $\partial Y_2 \neq \emptyset$ in X satisfying (o); moreover, ∂Y_1 has property (o). Assume that $Y_2 \subset X$ is a 2-dimensional compact connected C^0 -submanifold with $\partial Y_2 \neq \emptyset$ in X and ∂Y_1 and ∂Y_2 are isometric in the metrics ρ_{Y_j} ($j = 1, 2$). Then ∂Y_2 also has the property (o).*

This theorem has the following generalization.

Theorem 3.8 *Let $n = 2$. Suppose that (i) holds for a 2-dimensional closed connected C^0 -submanifold Y_1 with boundary $\partial Y_1 (\neq \emptyset)$ in a 2-dimensional smooth complete connected Riemannian manifold X without boundary satisfying (o). Assume that $Y_2 \subset X$ is a 2-dimensional closed connected C^0 -submanifold with $\partial Y_2 \neq \emptyset$ in X ; moreover, ∂Y_1 and ∂Y_2 are isometric in the metrics ρ_{Y_j} ($j = 1, 2$). Then Y_2 has the property (o) as well.*

Corollary 3.2 (of Theorem 3.7). *Assume that the hypothesis of Theorem 3.7 hold and that the manifold X has the following property: $\rho_X(x, y) = \rho_Y(x, y)$ for any two points x and y on the boundary ∂Y of every 2-dimensional compact connected C^0 -submanifold $Y \subset X$ with $\partial Y \neq \emptyset$ satisfying (i) and (o). Then ∂Y_1 and ∂Y_2 are isometric in the metric ρ_X of the ambient manifold X .*

Corollary 3.3 (of Theorem 3.8). *Assume that the hypothesis of Theorem 3.8 hold and that the manifold X has the following property: $\rho_X(x, y) = \rho_Y(x, y)$ for any two points x and y on the boundary ∂Y of every 2-dimensional closed connected C^0 -submanifold $Y \subset X$ with $\partial Y \neq \emptyset$ satisfying (i) and (o). Then ∂Y_1 and ∂Y_2 are isometric with respect to the metric ρ_X .*

Theorem 3.9 *Let $n \geq 3$. Suppose that (X, g) is an n -dimensional smooth connected Riemannian manifold without boundary and Y_1 and Y_2 are n -dimensional compact connected C^0 -submanifolds with nonempty boundaries*

∂Y_1 and ∂Y_2 in X satisfying conditions (i) and (ii) (in Theorem 3.5). Assume that Y_1 has property (o) and X satisfies the following condition: $\rho_X(x, y) = \rho_Y(x, y)$ for any two points x and y on the boundary ∂Y of every n -dimensional compact connected C^0 -submanifold $Y \subset X$ with $\partial Y \neq \emptyset$ satisfying (i), (ii), and (o). Suppose also that ∂Y_1 and ∂Y_2 are isometric in the metrics ρ_{Y_j} , where $j = 1, 2$. Then ∂Y_1 and ∂Y_2 are isometric in ρ_X .

Theorem 3.10 *Let $n \geq 3$. Suppose that (X, g) is an n -dimensional smooth complete connected Riemannian manifold without boundary and Y_1 and Y_2 are n -dimensional closed connected C^0 -submanifolds with nonempty boundaries ∂Y_1 and ∂Y_2 in X satisfying (i) and (ii). Assume that ∂Y_1 has property (o) and X satisfies the following condition: $\rho_X(x, y) = \rho_Y(x, y)$ for any two points x and y on the boundary ∂Y of every n -dimensional closed connected C^0 -submanifold Y with $\partial Y \neq \emptyset$ in X satisfying (i), (ii), and (o). Suppose also that ∂Y_1 and ∂Y_2 are isometric in the metrics ρ_{Y_j} ($j = 1, 2$). Then ∂Y_1 and ∂Y_2 are isometric in ρ_X .*

Proofs of Theorems 3.5 and 3.7-3.10 are similar to the proof of Theorem 3.4 (Theorems 3.5 and 3.7-3.10 can be proved using the corresponding analogs of Theorems 3.1 and 3.3).

4 On unique determination of domains by the condition of local isometry of boundaries in the relative metrics

THE CASE OF PLANE DOMAINS

The first main result of this Section is the following theorem.

Theorem 4.1 *Let U be a domain in \mathbb{R}^2 with smooth boundary. Then*

(i) if U is bounded, then it is uniquely determined in the class of all bounded plane domains with smooth boundaries by the condition of local isometry of boundaries in the relative metrics if and only if this domain is convex;

(ii) if U is unbounded, then the unique determination of U in the class of all plane domains with smooth boundaries by the condition of local isometry of boundaries in the relative metrics is equivalent to the strict convexity of this domain.

Remark 4.1. Let U be a domain in \mathbb{R}^n . As in [5], we say that U has smooth boundary, Lipschitz boundary if the Euclidean boundary ∂U of this

domain is an $(n-1)$ -submanifold of class C^1 (a Lipschitz submanifold) without boundary in \mathbb{R}^n . In the case of domain U with Lipschitz boundary, its Hausdorff boundary $\partial_H U$ is in natural way identified with Euclidean boundary and metric $\rho_{\partial U, U}$, corresponding to Hausdorff metric can be defined in the following manner:

$$\rho_{\partial U, U}(x, y) = \liminf_{x' \rightarrow x, y' \rightarrow y; x', y' \in U} \{\inf[l(\gamma_{x', y', U})]\},$$

where $x, y \in \partial U$ and $\inf[l(\gamma_{x', y', U})]$ is the infimum of lengths $l(\gamma_{x', y', U})$ of smooth paths $\gamma_{x', y', U} : [0, 1] \rightarrow U$ joining x' and y' in U . Recall also that a domain U is said to be strictly convex if it is convex and the interior of the segment joining any two points in its closure $\text{cl } U$ is contained in U (cf. with the hypothesis of Theorem 3.6).

Lemma 4.1 *Let U and V be two plane domains with smooth boundaries and $f : \partial U \rightarrow \partial V$ be a bijective mapping which is a local isometry of boundaries of these domains in the relative metrics. Then f is a (global) isometry of boundaries ∂U and ∂V in their intrinsic metrics.*

Lemma 4.2 *Suppose that domains U and V and mapping $f : \partial U \rightarrow \partial V$ satisfy to the hypothesis of Lemma 4.1, moreover, ∂U is bounded. Then the boundary ∂V of the domain V is also bounded and the mapping f has the following property: there exists a number $\varepsilon > 0$ such that $\rho_{\partial U, U}(a, b) = \rho_{\partial V, V}(f(a), f(b))$ if $a, b \in \partial U$ and $\rho_{\partial U, U}(a, b) < \varepsilon$.*

Lemma 4.3 *Under hypothesis of Lemma 4.1 and an additional supposition that the boundary ∂U of the domain U is connected, the boundary ∂V of the domain V is also connected.*

The proofs of these lemmas are sufficiently simple. By this reason, we omit them.

*Proof of Theorem 4.1. **Step 1.*** Prove the first part of assertion (i), i.e., prove that if U is a bounded convex plane domain with smooth boundary then it is uniquely determined in the class of all bounded plane domains with smooth boundaries by the condition of local isometry of boundaries in the relative metrics. To this end, suppose that for a bounded convex plane domain U with smooth boundary, there exists a bounded plane domain V with smooth boundary such that its boundary ∂V is locally isometric to the boundary ∂U of U in the relative metrics of boundaries (further, $f : \partial U \rightarrow \partial V$ is the fixed mapping realizing a such isometry). Then by

Lemmas 4.2 and 4.3, the boundary of the domain V is connected (and consequently, V is a Jordan domain), and f has the property indicated in Lemma 4.2. Accomplishing, if it is necessary, an additional inversion with respect to a straight line, we can also assume that $f : \partial U \rightarrow \partial V$ preserves the orientation of the boundary ∂U of U , induced by the canonical orientation of this domain, i.e., f "transfers" the mentioned orientation to the orientation of the boundary ∂V of V induced by the canonical orientation of V .

Let, further, $I = [a, b]$, where $a \neq b$, be a segment such that $I \subset \partial U$ and the image $f(I)$ of I is no longer a segment, moreover, any another segment $I^* = [a^*, b^*]$ ($a^* \neq b^*$, $I^* \subset \partial U$) of ∂U having common points with I is a subset of the segment I (below, we denote the set of all such segments I by the symbol Λ). We assert that $f(I)$ is a locally convex arc directed by its convexity inside the domain V . The latter means that every point $P \in f(I)$ has a closed neighborhood $N = N(P)$ for which $f(I) \cap N$ is a convex arc directed by its convexity inside V , i.e., $f(I) \cap N(P) = f(I_P)$, where $I_P = [\alpha_P, \beta_P] \subset I$, and the closed curve C_P , composed from $f(I_P)$ and the segment J_P joining the endpoints $f(\alpha_P)$ and $f(\beta_P)$ of the arc $f(I_P)$, either degenerates to the segment J_P , or is the boundary of a bounded convex domain with the following property. There is found a segment T with $\text{Int } T \subset V$, placed on straight line τ_P which is perpendicular to J_P and passes through its midpoint, moreover, some endpoint of T belongs to the arc $f(I_P)$ and its second endpoint is on the arc $(\partial V) \setminus f(I_P)$, both of these endpoints are on the same side from the straight line j_P containing the segment J_P , and the endpoint belonging to the arc $f(I_P)$ is nearer to j_P than the other endpoint. Assuming the contrary, i.e., supposing that $f(I)$ is no a locally convex arc directed by its convexity inside V , we (taking the smoothness of the boundary of V into account) arrive to the existence of a segment $I_P = [\alpha_P, \beta_P] \subset I$ such that either (1) $f(I_P)$ is a nonconvex arc, or (2) the arc $f(I_P)$ is no a segment and is a convex arc directed by its convexity inside the complement cV of V . In both cases, for the curve $f(\text{Int } I_P)$, there exist a point $Q \in f(\text{Int } I_P)$ and a locally supporting segment to this curve from the side of the complement cV of V all points of which, except the point Q , belong to the interior of cV and Q is the common point of this segment and the boundary ∂V of V . In the case of (2), these point and supporting segment can be found on the basis of the considerations using in the proof of the Theorem of Leja-Wilkosz [14] which is mentioned in [15], if we bring evident modifications corresponding to our case in it.

In the case of (1), the curve $f(\text{Int } I_P)$ contains a point L_P such that if we draw the tangent in L_P to our curve then there exist points $R_P \in f(\text{Int } I_P)$ and $S_P \in f(\text{Int } I_P)$ lying on various sides of this tangent. Replace the point

R_P , if it is necessary, by the point which is the nearest point to L_P on the segment $[R_P, L_P]$ (we will remain for this point the designation R_P) and belongs to the arc $f(\text{Int } I_P)$. Analogously, we replace the point S_P by the point of the arc $f(\text{Int } I_P)$ which is the nearest point to L_P on the segment $[L_P, S_P]$. And then consider two Jordan domains such that the boundary of the first of them is the union of the segment $[R_P, L_P]$ and of that arc from three arcs constituting the set $f(\text{Int } I_P) \setminus \{R_P, L_P\}$, the endpoints of which are R_P and L_P , and the boundary of the second domain is constructed by the same way on the basis of the points L_P and S_P and of the same arc $f(\text{Int } I_P)$. By the way of constructing, one of these domains will be contained in V and the second domain will be contained in cV . Considering the first domain of them and using the above-mentioned argument from the proof of theorem of Leja-Wilkosz in [15], it is not difficult to find the desired point Q on the part of its boundary disposed on $f(\text{Int } I_P)$, and a locally supporting segment j to the curve $f(\text{Int } I_P)$ at that point from the side of the complement cV of the domain V . Hence, in both cases (1) and (2), we arrived to desired situation. Transposing the tangent in the point Q to $f(\text{Int } I_P)$ in a parallel way to itself at a sufficiently short distance to it to the side where, so to say, the domain V lies, we will easily get the following situation: there exist three points R'_P , L_P and S'_P on the boundary ∂V of V belonging to the arc $f(\text{Int } I_P)$ and such that $]R'_P, S'_P[\subset V$, moreover, the segment $[R'_P, S'_P]$ cuts off from V the Jordan subdomain the boundary of which contains L_P . Clear that $f^{-1}(R'_P)$, $f^{-1}(L_P)$ and $f^{-1}(S'_P)$ are the successively ordered points on the interval $\text{Int } I_P$. Hence, the triangle inequality holds for these points in the metric $\rho_{\partial U, U}$, but by their choice, for the points R'_P , L_P and S'_P , the strict triangle inequality in the metric $\rho_{\partial V, V}$ holds. Since we could initially assume that the length of I_P is less than ε , where ε is the number from Lemma 4.2 corresponding to the mapping f which is considered now then we arrived to the contradiction because by this lemma, the equality in the triangle inequality must also be fulfilled for the points R'_P , L_P and S'_P . Therefore, $f(I)$ is a locally convex arc directed by its convexity inside V .

We assert that the set Λ is finite. Clear that by the finiteness of the length $l = l(\partial U)$ of the boundary ∂U of U , the finiteness of Λ follows from the fact that Λ does not contain segments the length of which does not exceed, for example, $\varepsilon/2$. Assuming that a segment $\Delta = [\alpha_\Delta, \beta_\Delta] \in \Lambda$ has the length $l(\Delta) \leq \varepsilon/2$, consider points Q and S of this segment such that $Q \neq S$, Q is situated nearer, let us say, to the left endpoint α_Δ of the segment, and $f(Q)$ and $f(S)$ lie on the same side (and at the positive distance) from the tangent τ to ∂V at the point $f(\alpha_\Delta)$, finally, the (least positive) angle between the tangent rays to the arcs $(\partial V) \setminus f(\Delta)$ and $f([\alpha_\Delta, S])$ in the points $f(\alpha_\Delta)$ and

$f(S)$, respectively, is less than $\pi/4$. Further, let a point $P \in (\partial U) \setminus \Delta$ is so close to α_Δ that $\rho_{\partial U, U}(P, \alpha_\Delta) < \varepsilon/2$ and the points $f(P)$ and $f(\alpha_\Delta)$ lie on the same side from each of the tangents to ∂V in the points $f(Q)$ and $f(S)$. Under these suppositions, for the triple of the points P , Q and S , the strict triangle inequality in the metric $\rho_{\partial U, U}$ holds, and for their images $f(P)$, $f(Q)$ and $f(S)$, the triangle equality (in the metric $\rho_{\partial V, V}$) holds. Thereby, by virtue of the choice of the number ε (and Lemma 4.2), we arrive to the contradiction from which it follows that $\Delta = \emptyset$ and consequently, Λ is finite.

Let $\omega : [0, l] \rightarrow \partial U$ be a natural parametrization of the boundary ∂U of U corresponding to the orientation of ∂U generated by the canonical orientation of the domain U , and let $[\alpha_1, \beta_1] \subset [0, l]$ and $[\alpha_2, \beta_2] \subset [0, l]$ be the segments such that $\omega([\alpha_j, \beta_j]) \in \Lambda$, where $j = 1, 2$, $\alpha_1 < \beta_1 < \alpha_2 < \beta_2$ and the arc $\omega(] \beta_1, \alpha_2])$ does not contain points of the segments from Λ . We assert that $f|_{\omega([\beta_1, \alpha_2])}$ is an Euclidean isometry (i.e., there exists an Euclidean isometry $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $F|_{\omega([\beta_1, \alpha_2])} = f|_{\omega([\beta_1, \alpha_2])}$). Indeed, if the arc $\omega([\beta_1, \alpha_2])$ does not contain segments, i.e., it is strictly convex, moreover, its convexity directed toward the interior of the complement cU of U . Hence, considering a point $c \in \omega(] \beta_1, \alpha_2])$ and sufficiently close to it points $a, b \in \omega(] \beta_1, \alpha_2])$, where $\beta_1 < \omega^{-1}(a) < \omega^{-1}(c) < \omega^{-1}(b) < \alpha_2$ (the closeness of the points a and b to the point c is such that the distance between each pair of the considering below triple of the points a , $f^{-1}(\gamma(s_0))$ and b is less than $\varepsilon/2$; we can easily obtain the latter using the hypothesis of theorem) and assuming that $[f(a), f(b)] \cap \text{Int}(cV) \neq \emptyset$, we arrive to a situation where for the shortest path $\gamma : [0, s] \rightarrow \text{cl } V$ joining the points $f(a)$ and $f(b)$ in the closure $\text{cl } V$ of the domain V ⁷, there exists a point $s_0 \in]0, s[$ for which $\gamma(s_0) \in \partial V$ and $f^{-1}(\gamma(s_0)) \in f^{-1}(\text{Im } \gamma \cap \partial V) \setminus \{a, b\} (\neq \emptyset)$. But then for the triple of the points a , $f^{-1}(\gamma(s_0))$ and b , the strict triangle inequality in the metric $\rho_{\partial U, U}$ holds, at the same time for the points $f(a)$, $\gamma(s_0)$ and $f(b)$, takes place the equality in the triangle inequality in the metric $\rho_{\partial V, V}$. Therefore, by virtue of Lemma 4.2, $[f(a), f(b)] \subset \text{cl } V$ from which the equality $|f(a) - f(b)| = |a - b|$ follows. Hence, the restriction $f|_{U_\varepsilon \cap \omega([\beta_1, \alpha_2])}$ of f to the intersection $U_\varepsilon \cap \omega([\beta_1, \alpha_2])$ of the ε -neighborhood $U_\varepsilon (= B(P, \varepsilon))$ of each point $P \in \omega([\beta_1, \alpha_2])$ and arc $\omega([\beta_1, \alpha_2])$ itself is an isometry in Euclidean metric. This circumstance allows easily to conclude that $f|_{\omega([\beta_1, \alpha_2])}$ is an Euclidean isometry. In the case where $\omega([\beta_1, \alpha_2])$ contains segments (which no longer belong to the set Λ and, consequently, their images under

⁷The existence of such shortest path is guaranteed, for instance, by the results of Section 3; see also [16].

the mapping f are also segments), the proof of the fact that $f|_{\omega([\beta_1, \alpha_2])}$ is an Euclidean isometry is close to the proof of this fact in the previous case, i.e., in the case of the strict convexity of $\omega([\beta_1, \alpha_2])$. The difference in the arguments consists of negligible and easily reproducible details, and we omit them.

Now, we are able to conclude the proof of the first part of assertion (i) of our theorem. If the boundary ∂U of U is such that $\Lambda = \emptyset$, then the first part of (i) is proved on the basis of the arguments from the previous item. In the case of $\Lambda \neq \emptyset$, consider a segment $\Delta \in \Lambda$ and accomplishing appropriate translation and rotation in the plane \mathbb{R}^2 , get the situation where the segment Δ lies on the ordinate axis, its upper endpoint is the origin, and the domain U is situated on the left half-plane. Let $\gamma : [0, l] \rightarrow \partial U$ ($\gamma(0) = \gamma(l) = (0, 0)$) be a natural parametrization of the boundary ∂U of U corresponding to the orientation of ∂U generated by the canonical orientation of U . If $f|_{\gamma([0, l-l(\Delta)])}$ is an Euclidean isometry then we can assume, without loss of generality, that $f|_{\gamma([0, l-l(\Delta)])} = \text{Id}_{\gamma([0, l-l(\Delta)])}$. Taking yet into account that $f(\gamma([l-l(\Delta), l])) = f(\Delta)$ is not a segment (because of $\Delta \in \Lambda$), we see that $f(\partial U) = \partial V$ is not a closed curve i.e., $f(\gamma(l)) \neq f(\gamma(0))$. The obtained contradiction leads us to the conclusion that $\Lambda = \emptyset$. Thus, in this case, the first part of (i) is proved. Further, assume that Λ consists of n segments $[\gamma(\alpha_1), \gamma(\beta_1)], [\gamma(\alpha_2), \gamma(\beta_2)], \dots, [\gamma(\alpha_n), \gamma(\beta_n)] = \gamma([l-l(\Delta), l]) = \Delta$, where $0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \dots < \alpha_n < \beta_n = l$. Since $f|_{\gamma([0, \alpha_1])}$ is an Euclidean isometry, we can assume, with loss of generality, that $f|_{\gamma([0, \alpha_1])} = \text{Id}_{\gamma([0, \alpha_1])}$. Then, using the induction argument, it is not difficult to show that the rotation of the vector ω , where $-\omega$ is the unit tangent vector to the curve $\gamma([l-l(\Delta), l])$ (i.e., to the segment Δ) in the point $\gamma(l)$, is realized (under the action of the mapping f) at the angle being equal to the following quantity:

$$V = - \sum_{k=1}^n \left\{ \sup_{\alpha_k \leq t_1 < t_2 < \dots < t_{\varkappa+1} \leq \beta_k} \sum_{\nu=1}^{\varkappa} |\theta\gamma(t_{\nu+1}) - \theta\gamma(t_{\nu})| \right\} \neq 0,$$

where $\theta\gamma(t)$ is the unit tangent vector to the curve $\gamma([t, l])$ in the point $\gamma(t)$ if $0 < t < l$ and to the curve $\gamma([0, l])$ in the point $\gamma(0) = (0, 0)$ when $t = l$. If $|V| < 2\pi$ then $\omega \neq \mu e_2$, where $\mu > 0$ and e_2 is the unit base vector of the ordinate axis. And if $|V| \geq 2\pi$ then (since f preserves the orientation of the boundary) without self-intersections, the curve $f(\partial V)$ can not be close. In both cases, we got the contradiction with the fact that the curve ∂V is closed and smooth. Hence, the first part of the assertion (i) of our theorem is completely proved.

Step 2. Prove the second part of assertion (i). Assuming that U is

a bounded nonconvex plane domain with smooth boundary, we will show that by the appropriate deformation, we can get another domain V whose boundary ∂V is smooth and locally isometric to the boundary ∂U of U in the relative metrics $\rho_{\partial U, U}$ and $\rho_{\partial V, V}$ of boundaries, and the domains U and V themselves are not isometric each other in Euclidean metric, i.e., there does not exist an Euclidean isometry $J : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $J(U) = V$. In the case where the boundary of U is not connected, a construction of the above-mentioned domain V realizes by a small permutation of a connected component of the boundary ∂U when the location of the other connected components leaves fixed. And if the boundary of U is connected, i.e., U is a Jordan domain, then we will argue in the following way. By theorem of Leja-Wilkosz [14], there will be found a "locally strict supporting outwards" segment I lying in U except a single interior point for I , let us say, point P , which belongs to ∂U . Consider a closed disk K centered at P and having so small radius r that the boundary circle of this disk intersects with I in two points and the interior of one of the half-disks K_+ and K_- such that $K_+ \cup K_- = K \setminus I$, for instance, $\text{Int } K_-$, does not contain points of the boundary ∂U of U . Let u and v are two straight lines which are perpendicular to I , situated on the various sides of the normal n to it at the point P , and sufficiently close to n . Let us consider the nearest points L and S of the sets $u \cap \partial U$ and $v \cap \partial U$ to the segment I and join them by the shortest in $\text{cl } U$ curve μ . Moreover, we regard r so small that the closure of the arc representing itself lesser of two arcs, which arise on the boundary ∂U when we remove the points L and S from it, is contained in $(\text{Int } K_+) \cup \{P\}$ and that (by the smoothness of ∂U) the curve μ is convex, smooth and directed by its convexity toward U . We can get one of two cases: (1) $\mu \subset \partial U$, and (2) μ contains segments interior of each of them is a subset of U . Further, consider (in both cases (1) and (2)) the points L^* and S^* belonging to $\lambda \cap \mu$ and chosen by the following way: L^* and L lie on the same side of both the straight line containing the segment I (moreover, the point L^* is situated nearer to this straight line than L) and the straight line ψ perpendicular to I , passing through the point P , besides, L^* is situated nearer to ψ than L , finally, the point S^* is defined in the similar way in comparison with the location of S . By symbol U^* , denote the Jordan domain with the boundary $((\partial U) \setminus \lambda^*) \cup \mu^*$, where λ^* and μ^* are the subarcs of the arcs λ and μ with the endpoints L^* and S^* .

In the case of (1), a necessary deformation of the domain $U = U^*$ realizes in the sufficiently obvious way and is reduced to a deformation of the curve μ^* . The arc μ^* replaces by a convex arc $\widetilde{\mu^*}$ of the same length, lying in the disk K and also directed by its convexity in U^* (more exactly, in the

new domain $\widetilde{U} = \widetilde{U}^*$, and the arc $(\partial U^*) \setminus \lambda^* = (\partial U^*) \setminus \mu^*$ leaves fixed, moreover, the closed arc $\widetilde{\mu}^* \cup ((\partial U^*) \setminus \mu^*)$ forms the smooth boundary of the new domain \widetilde{U}^* . It is not difficult to verify that the boundaries of U^* and \widetilde{U}^* are locally isometric in the relative metrics $\rho_{\partial U^*, U^*}$ and $\rho_{\partial \widetilde{U}^*, \widetilde{U}^*}$ (here, as a local isometry in the relative metrics of the boundaries of U^* and \widetilde{U}^* , we can take the mapping f of these boundaries which is leaving fixed the arc $(\partial U^*) \setminus \mu^*$). Clear also that in the process of the construction of our deformation, we can get the following situation: it is impossible to map the domain U^* onto the domain \widetilde{U}^* by an Euclidean isometry. Consequently, \widetilde{U}^* is the desired domain V .

In the case of (2), the construction of a new domain V realizes in the following way. If $\mu^* = \lambda^*$ then V is constructed as in the case (1). In the case where μ^* contains segments the interior of which lie in U and their endpoints belong to ∂U (denote the set of all such segments by the symbol \mathcal{M}^*), we, starting from the domain U^* , realize first the construction of the domain \widetilde{U}^* circumscribed in the case (1), but in addition, we will leave invariant the length of every segment of \mathcal{M}^* under the action of the arising (in the process of the construction) boundary mapping $f^* : \partial U^* \rightarrow \partial \widetilde{U}^*$. The latter is possible by the large degree of freedom in the construction of the curve $\partial \widetilde{U}^*$ which is submitted by the condition to that the curve μ^* satisfies in the case (2)⁸. In this case, the final mapping $f : \partial U \rightarrow \partial \widetilde{U}$, where \widetilde{U} ($= V$) is a desired new domain, is constructed like this: it leaves fixed the curve $(\partial U) \setminus \lambda$ and coincides with f^* on the set $N = \mu^* \cap \lambda^*$. And if the arc χ with endpoints A and B has not common points with $(\partial U) \setminus \lambda^*$ and is cut off from ∂U by a segment from \mathcal{M}^* , then we subject this curve to the action of the preserving orientation Euclidean isometry $J : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $J(A) = f^*(A)$ and $J(B) = f^*(B)$, and then put $f|_\chi = J|_\chi$. In this case, the domain V is the Jordan domain with the boundary $f(\partial U)$ and by the construction, $f : \partial U \rightarrow \partial V$ is a local isometry of the boundaries of U and V in their relative metrics, moreover, the large degree of freedom in the choice of the above-circumscribed deformation of the domain U which still takes place, makes possible to realize this deformation such that the domains U and V are not isometric in the Euclidean metric. So, in both cases (1) and (2), we get the following situation: if U is nonconvex bounded plane domain with smooth boundary then it is not uniquely determined in the class of all bounded plane domains with smooth boundaries by the condition of local isometry of boundaries in the relative metrics. Consequently, the assertion (i) of the theorem is completely proved.

⁸In this connection, see Lemma 5.1.

Step 3. Pass to prove the assertion (ii). The fact, that an unbounded strictly convex plane domain U with smooth boundary is uniquely determined in the class of all plane domains with smooth boundaries by the condition of local isometry of boundaries in the relative metrics, can be proved on the basis of the arguments from the proof of the first part of assertion (i). Considering one more plane domain V with smooth boundary and assuming that the boundaries of the domains U and V are locally isometric in their relative metrics and modify negligibly the arguments from the proof of the assertion (i), we establish that ∂U and ∂V are isometric in the Euclidean metric from where the isometry of the domains U and V themselves follows.

Step 4. Proving the second part of assertion (ii), at first, we make sure that if an unbounded plane domain U with smooth boundary is not convex then by the same method as we used in the proof of the second part of assertion (i), it can be deformed to a domain V with smooth boundary, moreover, to such domain that the boundaries ∂U and ∂V are found to be locally isometric in their relative metrics, and for these domains themselves, there does not exist an Euclidean isometry $J : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with property $V = J(U)$. In the considering case, there exists a deformation of the boundary ∂U of the domain U which does not lead us to the desired result, but the above-mentioned degree of freedom in a choice of a deformation makes possible to pass easily over this difficulty.

Step 5. Now, let U be an unbounded plane convex domain with smooth boundary which is not strictly convex. In this case, a construction of an above-mentioned domain V achieves by quite simple methods. Indeed, on the boundary ∂U of our domain, there exists a segment, let us say, I . We can assume that any other segment having common points with I and lying on ∂U , is a subset of I . Without loss of generality, we will also suppose that I is the segment of the abscissa axis with the endpoints $A = (-2l, 0)$ and $B = (2l, 0)$ and the domain U is found in the lower half-plane. Subject the boundary ∂U of the domain to the following transformation. The origin divides the boundary for two curves. The curve from those curves, which contains the segment with endpoints $(0, 0)$ and $(0, 2l)$, leaves under this transformation fixed. The segment with endpoints $(-l, 0)$ and $(0, 0)$ is transformed to the quarter of the circle $\{(x, y) \in \mathbb{R}^2 : x^2 + (y - \frac{2l}{\pi})^2 = \frac{4l^2}{\pi^2}\}$ with endpoints $(0, 0)$ and $P = (-\frac{2l}{\pi}, \frac{2l}{\pi})$. The remaining part of the boundary ∂U is first subjected to the translation at vector $((1 - \frac{2}{\pi})l, \frac{2l}{\pi})$ and then to the rotation at angle $-\frac{\pi}{2}$ with respect to P . As the final result, we get the curve γ dividing the plane on two unbounded domains. That domain from them which locally adjoins from below to the segment with endpoints

$(0,0)$ and $(0,2l)$, we will take for a domain V . Easily to verify that the boundary ∂V of this domain is locally isometric to the boundary ∂U of the initial domain U in the relative metrics $\rho_{\partial U, U}$ and $\rho_{\partial V, V}$ of boundaries, and the domains U and V themselves are not isometric to each other in Euclidean metric. Thus, assertion (ii), and together with it, Theorem 4.1 are completely proved.

In connection with Theorem 4.1, it should be noted that there exists a bounded plane domain U with smooth boundary which is not uniquely determined in the class of all plane domains with smooth boundaries by the condition of local isometry of boundaries in the relative metrics (see [17]).

In the case where the boundary of a domain $U \subset \mathbb{R}^2$ is not smooth, Theorem 4.1 ceases to be valid. Really, the following assertion is correct.

Theorem 4.2 *There exists a bounded plane domain U with Lipschitz boundary such that it is not convex but, at the same time, is uniquely determined in the class of all plane domains by the condition of local isometry of boundaries in the relative metrics.*

Remark 4.2. Theorem 4.2 due to M. V. Korobkov (see [18]). An argument of its proof will be discussed below.

UNIQUE DETERMINATION OF SPACE DOMAINS

Now, consider the case of space domains. Below, we will use the following assertion which is a generalization of Lemma 2.3 to the case of locally isometric mappings of the boundaries of domains.

Lemma 4.4 *Let U, V be domains in space \mathbb{R}^n ($n \geq 2$) such that there exists a bijective mapping $f : \partial_H U \rightarrow \partial_H V$ local isometric in the relative metrics of their Hausdorff boundaries. Then for every element $w \in \partial_H U$, there exists a number $\varepsilon = \varepsilon_w > 0$ satisfying the following condition: for any two elements $a', b' \in \partial U$ such that $]a', b'[\subset U$ and the elements $a, b \in \partial_H U$ generated by the path $\gamma(t) = tb' + (1-t)a'$, $t \in [0, 1]$ (i.e., generated by the Cauchy sequences in the intrinsic metric ρ_U of the domain U $\{\gamma(1/n)\}_{n=3,4,\dots}$ and $\{\gamma(1-1/n)\}_{n=3,4,\dots}$, respectively), belong to the ε -neighborhood $Z(w) = \{z \in \partial_H U : \rho_{\partial_H U, U}(z, w) < \varepsilon\}$ of the element w , the relation $]p_V f(a), p_V f(b)[\subset V$ holds.*

The proof of this lemma differs from the proof of Lemma 2.3 by negligible modifications, therefore, we omit it.

Now, suppose that a considering domain is strictly convex. Then the following theorem is valid.

Theorem 4.3 *Let $n \geq 2$. If a domain U in space \mathbb{R}^n is strictly convex, then it is uniquely determined in the class of all domains in this space by the condition of local isometry of boundaries in the relative metrics.*

Proof. Let V be a domain such that there exists a bijective mapping $f : \partial_H U \rightarrow \partial_H V$ being a local isometry in the relative metrics of the Hausdorff boundaries $\partial_H U$ and $\partial_H V$ of the domains U and V . Assume that x and y are points of the Euclidean boundary ∂U of the domain U (by the strict convexity of U and Remark 4.1, we can suppose that x and y are simultaneously elements of the Hausdorff boundary $\partial_H U$ of U). By Lemma 4.4, each element $w \in \partial_H U$ has an ε_w -neighborhood $Z(w) = \{z \in \partial_H U : \rho_{\partial_H U, U}(z, w) < \varepsilon\}$ with the property: for any points $a, b \in Z(w)$ the relation $]p_V f(a), p_V f(b)[\subset V$ holds (as for $Z(w)$, see Lemma 4.4). From this fact, it follows that the mapping $\bar{f} : \partial U \rightarrow \partial V$ such that $\bar{f}(x) = p_V f(x)$ if $x \in \partial U$ is locally isometric in Euclidean metric (i.e., if $w \in \partial U$, then for each point $z \in Z(w)$, there exist a ball $B_x = B(x, r_x) \subset \mathbb{R}^n$ and an isometric mapping $F_x : \mathbb{R}^n \rightarrow \mathbb{R}^n$ in the Euclidean metric such that $F_x|_{(\partial U) \cup B_x} = \bar{f}|_{(\partial U) \cup B_x}$).

Let $\bar{f}(\partial U) = T \subset \partial V$. We assert that the closure $\text{cl} T$ of the set T coincides with the Euclidean boundary ∂V of V . Assuming that $M = ((\partial V) \setminus \text{cl} T) \neq \emptyset$, consider a point $z \in M$. Since $\text{cl} T$ is a closed set then $\text{dist}\{z, T\} = \text{dist}\{z, \text{cl} T\} > 0$. Taking yet into account that by Lemma 2.1, the set of supports of the Hausdorff boundary of a domain is dense on its Euclidean boundary, we can assert the existence of an element a of the Hausdorff boundary $\partial_H V$ such that its support $a' = p_V a$ satisfies to the condition $\text{dist}\{a', T\} = \text{dist}\{a', \text{cl} T\} > 0$. Let $\tilde{a} = f^{-1}(a)$. We have $\bar{f}(\tilde{a}) = p_V f(\tilde{a}) = p_V(f(f^{-1}(a)) = p_V a = a' \in T$. Therefore, $\text{cl} T = \partial V$.

Further, show that the mapping \bar{f} can be extended to an Euclidean isometry $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ of all space \mathbb{R}^n . Indeed, let a and b be any two points on the Euclidean boundary ∂U of U . We will now establish that

$$|\bar{f}(a) - \bar{f}(b)| = |a - b|. \quad (4.1)$$

To this end, consider a path $\gamma : [0, 1] \rightarrow \partial U$ the endpoints of which are $\gamma(0) = a$ and $\gamma(1) = b$. Since \bar{f} is a local isometry in the Euclidean metric then for each point $t \in [0, 1]$, we can find a ball $B_t = B(\bar{f}(\gamma(t)), r_t) \subset \mathbb{R}^n$ such that there exists an isometric in the Euclidean metric mapping $F_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with the property $F_t|_{(\partial U) \cap B_t} = \bar{f}|_{(\partial U) \cap B_t}$. By the continuity of the path γ , the sets $\gamma^{-1}((\partial U) \cap B_t)$, where $t \in [0, 1]$, form a covering of the segment $[0, 1]$ which is open with respect to $[0, 1]$. But then we can extract a finite subcovering $\{E_s = \gamma^{-1}((\partial U) \cap B_{t_s}), s = 1, \dots, k\}$. If $E_{s_1} \cap E_{s_2} \neq \emptyset$

where $1 \leq s_1, s_2 \leq k$ then $(\partial U) \cap B_{t_{s_1}} \cap B_{t_{s_2}} \neq \emptyset$. Taking into account the strict convexity of the domain U , we easily conclude that $F_{t_{s_1}} = F_{t_{s_2}}$. Thereby, we can assert that there exists the single isometric in the Euclidean metric mapping $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $F_s = F$ for all $s = 1, \dots, k$ and, consequently, $\bar{f}|_{\text{Im } \gamma} = F|_{\text{Im } \gamma}$. The latter implies the desired equality (4.1). And from it, by its turn (with regard for the above-stated), the assertion of the theorem follows.

Remark 4.3. Theorem 4.3 is a generalization of a theorem of A. D. Aleksandrov about the unique determination of the boundary ∂U of a strictly convex domain $U \subset \mathbb{R}^n$ by the relative metric $\rho_{\partial U, U}$ (see Theorem 3.6).

Proof of Theorem 4.2. Assume that U is a bounded nonconvex domain in \mathbb{R}^2 with Lipschitz boundary ∂U , and there exists such point $P \in \partial U$ that on the set $\partial U \setminus \{P\}$, the domain U is locally strictly convex, moreover, its convexity directed to the complement cU of this domain. We assert that U is uniquely determined by the condition of local isometry of boundaries in the relative metrics. The proof of this assertion realizes by the same scheme and with using the same tools as in the proof of Theorem 4.3, with certain negligible modifications. We turn our attention to them briefly.

In the considering now case let V be a one more domain in \mathbb{R}^2 whose Hausdorff boundary is locally isometric to the Hausdorff boundary of U , let $f : \partial_H U \rightarrow \partial_H V$ be a bijection which is a local isometry in the relative metrics of the Hausdorff boundaries $\partial_H U$ and $\partial_H V$ of U and V , finally, let $T = \bar{f}((\partial U) \setminus \{P\})$ (since the boundary ∂U of U is Lipschitz, we, taking into account Remark 4.1, identify $\partial_H U$ with ∂U). We assert that in this case, just as in the proof of Theorem 4.3, $\text{cl } T = \partial V$. The latter can be proved on the basis of the arguments from the proof of Theorem 4.3. Nevertheless, by Lemma 2.1 and the infinity of that part of the set of supports of the Hausdorff boundary $\partial_H V$ which is contained in the set $M = (\partial V) \setminus \text{cl } T$, the indicated there point a' can be chosen so that $\alpha = \bar{f}^{-1}(a) (\neq \emptyset) \subset (\partial U) \setminus \{P\}$.

The further arguments iterate the arguments used in the proof of Theorem 4.3 almost verbatim. By this reason, we omit them.

As opposed to that what takes place in the case of domains in \mathbb{R}^2 (see Theorem 4.1), in the case of space domains, under the decision of problems on their unique determination by the condition of local isometry of boundaries in the relative metrics, the condition of convexity of a considerable domain (as in Theorem 4.2) ceases to be necessary. In fact, the following assertion holds.

Theorem 4.4 *In \mathbb{R}^3 , there exists a domain U with smooth boundary such*

that it is uniquely determined in the class of all three-dimensional domains with smooth boundaries by the condition of local isometry of boundaries in the relative metrics but is not convex.

Proof. Let our domain U be made by the following way.
Consider the arc of cardioid

$$\theta = \{(x, y, z) \in \mathbb{R}^3 : x^2 + z^2 - \sqrt{x^2 + z^2} + z = 0, x^2 + z^2 > 0, x \geq 0, y = 0\}.$$

Leaving it fixed except of the part θ_1 which is cut out from it by the disk $\{(x, y, z) \in \mathbb{R}^3 : x^2 + z^2 \leq \frac{1}{9}, y = 0\}$, replace the arc θ_1 of the cardioid by the arc of the circle $\{(x, y, z) \in \mathbb{R}^3 : z = 1 - \sqrt{\frac{2}{3} - x^2}, 0 \leq x \leq \frac{\sqrt{5}}{9}, y = 0\}$. It is not difficult to verify that under the rotation of the curve obtained on this way around the axis Oz (up to the completed rotation), we obtain the closed smooth surface being the boundary of a three-dimensional nonconvex Jordan domain which we will accept for the desired domain U , establishing further that it is uniquely determined in the class of all domains in \mathbb{R}^3 with smooth boundaries by the condition of local isometry of boundaries in the relative metrics.

So, let $V \subset \mathbb{R}^3$ be another domain with smooth boundary and $f : \partial U \rightarrow \partial V$ be a bijective mapping of the boundary ∂U of U onto the boundary ∂V of V which is a local isometry of the boundaries ∂U and ∂V in their relative metrics. Consider the curve $\theta_0 = \theta \setminus \{(x, y, z) \in \mathbb{R}^3 : x^2 + z^2 \leq 1/4, y = 0\}$. Under the rotation around the axis Oz , this part of cardioid forms a locally strictly convex region S of the boundary of U directed by its convexity in the complement cU of this domain. Applying the same technique as in the proof of Theorem 4.3 and being based on Lemma 4.4 in addition, we first see for ourselves that there exists an isometry $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ in the Euclidean metric such that $f|_S = F|_S$.

Without loss of generality, we can assume that $F = \text{Id}_{\mathbb{R}^3}$. Suppose that S^* is the part of the boundary ∂U of U obtained under the rotation of the arc $\theta^* = \text{cl}(\theta \setminus (\theta_1 \cup \theta_0))$, and consider the intersection of S^* with a closed half-plane for which the axis Oz is the boundary. We can also assume that this intersection is the curve θ^* . Now, we show that any two sufficiently close points a and b of this curve (note that the degree of closeness of these points is determined by Lemma 4.2 in application to the mapping f) cut out from it an arc ab the image of which under the mapping f is a plane curve. Indeed, considering the third point c of the arc ab , taking into account the local strict convexity of the arc θ^* (with respect to the plane domain $U_{x,z} = U \cap \{(x, y, z) \in \mathbb{R}^3 : y = 0\}$, moreover, by its convexity in the

plane $\tau_{x,y} = \{(x, y, z) \in \mathbb{R}^3 : y = 0\}$, this arc is directed to the side of the complement $\tau_{x,y} \setminus U_{x,z}$ of $U_{x,z}$, and applying Lemma 4.4 to each pair of the triple of points a , c and b , we come to the conclusion that the point $f(c)$ is on the surface \tilde{S} formed by the rotation of the points of the arc $f(ab) = f(a)f(b)$ around the straight line ζ passing through the points $f(a)$ and $f(b)$, and the intersection of \tilde{S} with each half-plane, whose boundary is ζ , has the same length as the arc ab . If we suppose that the arc $f(ab)$ is not plane then its length will be greater than the length of the arc ab . The latter contradicts to Lemma 4.1. Hence, the arc $f(ab)$ is plane. Making arguments close to those which is used in the proof of the first part of assertion (ii) of Theorem 4.1, we establish further the existence of an isometry $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ in the Euclidean metric such that $F|_{ab} = f|_{ab}$. Therefore, the arc $f(ab)$ (together with the arc ab) is strictly convex and, consequently, if two planes contain the arc $f(ab)$ then they coincide. Extending our last considerations to the arc $\theta^* \cup \theta_0$, taking into account the above-said, and using the induction argument, it is not difficult to establish that the curve $f(\theta^* \cup \theta_0)$ is contained in the plane $\tau_{x,y}$, i.e., in the same plane that the curve $\theta^* \cup \theta_0$. Using again the arguments from the proof of Theorem 4.1, we come to the assertion that $f|_{\theta^* \cup \theta_0} = \text{Id}_{\theta^* \cup \theta_0}$. Considering the rest intersections of the domain U with half-planes whose border is axis Oz and taking into account all above-stated, we obtain as the result such relation

$$f|_W = \text{Id}_W$$

where W is the part of the boundary ∂U of U which is obtained by the rotation of the arc $\theta^* \cup \theta_0$ around the axis Oz .

Assume that $M = f((\partial U) \setminus W) \cap cV \cap \{(x, y, z) \in \mathbb{R}^3 : z \geq \frac{2}{9}\} \neq \emptyset$. Let $\alpha > \frac{2}{9}$ and such that

$$M_\alpha = M \cap \{(x, y, z) \in \mathbb{R}^3 : z = \alpha\} \neq \emptyset$$

and

$$M \cap \{(x, y, z) \in \mathbb{R}^3 : z > \alpha\} = \emptyset.$$

Suppose that, in M_α , there exists a point $(\bar{x}, \bar{y}, \alpha)$ such that $\bar{x}^2 + \bar{y}^2 > 0$. Without loss of generality, we can set that $\bar{x}^2 + \bar{y}^2 = \max_{(x,y,z) \in M_\alpha} (x^2 + y^2)$.

Besides, since $M_\alpha \cap f(W) = \emptyset$ then $(\bar{x}, \bar{y}, \alpha) \notin f(W)$. Further, let $\chi = \{\bar{x}(1 + \lambda/\sqrt{\bar{x}^2 + \bar{y}^2})e_1 + \bar{y}(1 + \lambda/\sqrt{\bar{x}^2 + \bar{y}^2})e_2 + (\alpha - \lambda t)e_3 : \lambda \geq 0\}$ be a ray outgoing from the point $P_0 = (\bar{x}, \bar{y}, \alpha)$, moreover, the value of the parameter t (> 0) is so small that this ray intersects $f((\partial U) \setminus W) \setminus \{P_0\}$ and the distance between P_0 and the nearest point P of the set $(f((\partial U) \setminus W) \setminus \{P_0\}) \cap \chi$ to

it is lesser than the number $\varepsilon = \varepsilon_{P_0}$ from Lemma 4.4 for the mapping f^{-1} (in this connection, note that the plane $\tau_\alpha = \{(x, y, z) \in \mathbb{R}^3 : z = \alpha\}$ is supporting to the surface $f((\partial U) \setminus W)$ and, therefore, is a tangent plane to it in all points $R \in M_\alpha$). Consequently, by the lemma and the fact that the interval $]P, P_0[$ is contained in V , the interval $]f^{-1}(P), f^{-1}(P_0)[$ must be contained in U . But this is impossible. Therefore, it remains to consider the case of $\bar{x} = \bar{y} = 0$. And in this case, we also have the contradiction, considering, for example, the ray $\{\lambda e_1 + (\alpha - \lambda t)e_3 : \lambda \geq 0\}$ as a desired ray, and further, repeating the arguments used in the previous case.

We must yet discuss the case $\alpha = \frac{2}{9}$. If $\text{dist}(M \cap \tau_{\frac{2}{9}}, W) > 0$ then using the arguments from the previous two cases, we see that this situation is also impossible. Now, let $\text{dist}(M \cap \tau_{\frac{2}{9}}, W) = 0$. The stated above facts and the smoothness of the boundaries ∂U and ∂V of U and V imply the following circumstance: for each point $z^0 \in M_{\frac{2}{9}} (= M \cap \tau_{\frac{2}{9}})$, there exists a number $\varkappa_0 > 0$ such that any ray emitted from z^0 and intersecting the cone $K = \{(x, y, z) \in \mathbb{R}^3 : z = \frac{1}{7} + \frac{5}{63}\sqrt{x^2 + y^2}, \frac{1}{7} \leq z \leq \frac{2}{9}\}$ in a point contained between the planes $\tau_{\frac{2}{9}}$ and $\tau_{\frac{2}{9} - \varkappa_0}$, has common points with the surface $(f((\partial U) \setminus W)) \setminus \{z^0\}$ (here we take into account that the generators of the cone K pass through the points of the boundary of the manifold $\text{cl}((\partial U) \setminus W)$, being tangent in these points to the boundary ∂V of V). Choosing as z^0 a point which is so near to W that the segment $[z^0, \tilde{z}]$ (where $\tilde{z} \in K \cap \tau_{\frac{2}{9} - \frac{\varkappa_0}{2}}$) of the ray χ emitted from it and intersecting with the circle $K \cap \tau_{\frac{2}{9} - \frac{\varkappa_0}{2}}$, has the least of possible lengths of such segments, consider the nearest point $P \in (f((\partial U) \setminus W) \setminus \{z^0\}) \cap \chi$ to the point z^0 . Setting in addition that $|P - z^0| < \varepsilon_{z^0}$ (where ε_{z^0} is a number for the mapping f^{-1} from Lemma 4.4), we can apply the above-mentioned arguments to make sure that this case is also impossible. At the final result, we have the inequality

$$f_3(x, y, z) < \frac{2}{9} \quad (4.2)$$

(where $f = (f_1, f_2, f_3) : \partial U \rightarrow \partial V$), which holds for all points $(x, y, z) \in (\partial U) \setminus W$.

Consider the bounded open set $A \in \mathbb{R}^3$ whose boundary is composed from the sets $f((\partial U) \setminus W)$ and $\Xi = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq \frac{5}{81}, z = \frac{2}{9}\}$. It is the three-dimensional Jordan domain contained in the complement to V . Now, we will prove that the domain A is convex. Assume by contradiction that this is not valid. Using the proof of theorem of Leja-Wilkosz [14] that is set forth in [15], we can assert the existence of three points $X \in \text{Int } A$, $Y \in \text{Int } A$ and $Z \in \text{Int } A$ such that $[X, Y] \subset \text{Int } A$,

$[Y, Z] \subset \text{Int } A$, $[X, Z] \not\subset \text{Int } A$, starting from which and fixing the location of plane τ containing these points, we can construct in this plane, for instance, a locally supporting outwards A concave arc of ellipse γ . And then changing a location of the point Z in its small spherical neighborhood, we can obtain a continual family of locally supporting outwards concave arcs of ellipses. The plane measure of each part of the boundary ∂V of V which is found in one of the indicated plane intersections, can not be positive, since ∂V is a smooth bounded surface and, consequently, has a finite area. Therefore, there exist segments $[a, b]$ of arbitrary small linear sizes such that $]a, b[\subset cA$ and $a, b \in \partial A$, moreover, we can also assume that $a, b \notin \Xi$. Hence, we are again found that we have the above-discussed situation in the process of proving of relation (4.2) from which it follows that the domain A is convex.

As the final result of our arguments for the surfaces $\text{cl}((\partial U) \setminus W)$ and $f(\text{cl}((\partial U) \setminus W))$, we are found themselves in the situation of theorem 2 of Section 7 from Chapter 3 of monograph of A. V. Pogorelov [1]. Using it, we see that these surfaces are equal. Taking into account the latter and also stated above facts in the process of proving, we can assert that our theorem is completely proved.

5 Appendix

Lemma 5.1 *Let $f_1 : [0, a^*] \rightarrow \mathbb{R}$ ($a^* > 0$) be convex downwards strictly increasing smooth function such that $f_1(0) = f_1'(0) = 0$, moreover, the graph Γ_1 of this function contains straight line segments, the union of the set \mathcal{M} of all such segments is dense in Γ_1 and $(0, 0)$ and $(a^*, f_1(a^*))$ are limit points for the set of the left endpoints of the segments from \mathcal{M} (we assume that the segments $\Delta \in \mathcal{M}$ are maximal in such sense that any segment $\tilde{\Delta} \subset \Gamma_1$ containing Δ coincides with it). Then for each number $\varepsilon > 0$, there exists a convex downwards strictly increasing smooth function $f_2 : [0, a^*] \rightarrow \mathbb{R}$ differing from f_1 and having the following properties: $\|f_2 - f_1\|_{C([0, a^*])} \leq \varepsilon$, $f_2(0) = f_2'(0) = 0$, $f_2(a^*) = f_1(a^*)$, $f_2'(a^*) = f_1'(a^*)$ and the mapping $F : \Gamma_1 \rightarrow \Gamma_2$ of the graphs of the functions f_1 and f_2 defined by the formula*

$$F : (x, y) \mapsto (\varphi^{-1}(x), f_2(\varphi^{-1}(f_1^{-1}(y)))) \in \Gamma_2, \quad (x, y) \in \Gamma_1,$$

where $\varphi : [0, a^*] \rightarrow [0, a^*]$ is the diffeomorphic solution of the functional equation

$$\int_0^{\varphi(x)} \{1 + [f_1'(\varphi)]^2\}^{1/2} d\varphi = \int_0^x \{1 + [f_2'(t)]^2\}^{1/2} dt, \quad 0 \leq x \leq a^*,$$

is isometric in the intrinsic metrics of the curves Γ_1 and Γ_2 which transforms each straight line segment of Γ_1 to a straight line segment of Γ_2 with the same length.

Proof. Let x_1, x_2 and x_3 be three points of the interval $]0, a^*[$ such that $x_1 < x_2 < x_3$ and these points are the left endpoints of the segments from the set \mathcal{M} (the choice of the points x_1, x_2, x_3 will be made more precise below). Assume that k_1, k_2, k_3 and k_4 are four real positive numbers. We will choose the function f_2 among functions having the following form:

$$f_2(x) = \begin{cases} k_1 f_1(x), & 0 \leq x < x_1; \\ (k_1 - k_2)[f_1(x_1) + f'_1(x_1)(x - x_1)] + k_2 f_1(x), & x_1 \leq x < x_2; \\ \sum_{s=1}^2 (k_s - k_{s+1})[f_1(x_s) + f'_1(x_s)(x - x_s)] + k_3 f_1(x), & x_2 \leq x < x_3; \\ \sum_{s=1}^3 (k_s - k_{s+1})[f_1(x_s) + f'_1(x_s)(x - x_s)] + k_4 f_1(x), & x_3 \leq x \leq a^*. \end{cases}$$

The equalities $f_2(a^*) = f_1(a^*)$ and $f'_2(a^*) = f'_1(a^*)$ leads us to the conditions

$$\sum_{s=1}^3 (k_s - k_{s+1})[f_1(x_s) + f'_1(x_s)(a^* - x_s)] + (k_4 - 1)f_1(a^*) = 0 \quad (5.1)$$

and

$$\sum_{s=1}^3 (k_s - k_{s+1})f'_1(x_s) + (k_4 - 1)f'_1(a^*) = 0. \quad (5.2)$$

The last condition will be result of the demand $\varphi(a^*) = a^*$. And since this demand is equality

$$\int_0^{a^*} \{1 + [f'_1(t)]^2\}^{1/2} dt = \int_0^{a^*} \{1 + [f'_2(t)]^2\}^{1/2} dt$$

then we have

$$\begin{aligned} & \int_0^{a^*} \{1 + [f'_1(t)]^2\}^{1/2} - \\ & \sum_{j=0}^3 \int_{x_j}^{x_{j+1}} \left\{ 1 + \left[\sum_{s=1}^j (k_s - k_{s+1})f'_1(x_s) + k_{j+1}f'_1(t) \right]^2 \right\}^{1/2} dt = 0, \quad (5.3) \end{aligned}$$

where $x_0 = 0$, $x_4 = a^*$ and $\sum_{s=1}^0 \dots = 0$.

The element $(k_1, k_2, k_3, k_4) = (1, 1, 1, 1) \in \mathbb{R}^4$ is a solution to the system (5.1)-(5.3). At the same time, by the construction, each straight line segment Δ of the curve Γ_1 is transformed to a straight line segment of the curve Γ_2 , moreover, $l(F(\Delta)) = l(\Delta)$. Now, it is sufficient to prove that the rank of the Jacobi matrix of the left parts of the equalities (5.1)-(5.3) calculated with respect to the variables k_1, k_2, k_3 and k_4 in the point $(1, 1, 1, 1)$ is equal to 3 under the successful choice of x_1, x_2 and x_3 .

To this end, represent the mentioned matrix in the following form:

$$N = (A_{js}) \quad \begin{matrix} j = 1, 2, 3 \\ s = 1, 2, 3, 4 \end{matrix}, \quad (5.4)$$

where

$$\begin{aligned} A_{11} &= -u_1 - f'_1(x_1) = - \int_0^{x_1} \frac{[f'_1(t)]^2 dt}{\{1 + [f'_1(t)]^2\}^{1/2}} - f'_1(x_1) \int_{x_1}^{a^*} \frac{f'_1(t) dt}{\{1 + [f'_1(t)]^2\}^{1/2}}, \\ A_{12} &= - \int_{x_1}^{x_2} \frac{f'_1(t)[f'_1(t) - f'_1(x_1)] dt}{\{1 + [f'_1(t)]^2\}^{1/2}} - \int_{x_2}^{a^*} \frac{f'_1(t)[f'_1(x_2) - f'_1(x_1)] dt}{\{1 + [f'_1(t)]^2\}^{1/2}}, \\ A_{13} &= - \int_{x_2}^{x_3} \frac{f'_1(t)[f'_1(t) - f'_1(x_2)] dt}{\{1 + [f'_1(t)]^2\}^{1/2}} - \int_{x_3}^{a^*} \frac{f'_1(t)[f'_1(x_3) - f'_1(x_2)] dt}{\{1 + [f'_1(t)]^2\}^{1/2}}, \\ A_{14} &= - \int_{x_3}^{a^*} \frac{f'_1(t)[f'_1(t) - f'_1(x_3)] dt}{\{1 + [f'_1(t)]^2\}^{1/2}}, \\ A_{21} &= f_1(x_1) + (a^* - x_1)f'_1(x_1), \\ A_{22} &= f_1(x_2) - f_1(x_1) + (a^* - x_2)f'_1(x_2) - (a^* - x_1)f'_1(x_1), \\ A_{23} &= f_1(x_3) - f_1(x_2) + (a^* - x_3)f'_1(x_3) - (a^* - x_2)f'_1(x_2), \\ A_{24} &= f_1(a^*) - f_1(x_3) - (a^* - x_3)f'_1(x_3), \quad A_{31} = f'_1(x_1), \\ A_{32} &= f'_1(x_2) - f'_1(x_1), \quad A_{33} = f'_1(x_3) - f'_1(x_2), \quad A_{34} = f'_1(a^*) - f'_1(x_3). \end{aligned}$$

The rank of the matrix (5.4) coincides with the rank of the matrix

$$\tilde{N} = \left(\sum_{\nu=1}^s A_{j\nu} \right) \quad \begin{matrix} j = 1, 2, 3 \\ s = 1, 2, 3, 4 \end{matrix},$$

in which

$$\begin{aligned} \sum_{\nu=1}^2 A_{1\nu} &= -u_2 - f'_1(x_2)v_2 = \\ &= - \int_0^{x_2} \frac{[f'_1(t)]^2 dt}{\{1 + [f'_1(t)]^2\}^{1/2}} - f'_1(x_2) \int_{x_2}^{a^*} \frac{f'_1(t) dt}{\{1 + [f'_1(t)]^2\}^{1/2}}, \end{aligned}$$

$$\begin{aligned} \sum_{\nu=1}^3 A_{1\nu} &= -u_3 - f'_1(x_3)v_3 = \\ &= - \int_0^{x_3} \frac{[f'_1(t)]^2 dt}{\{1 + [f'_1(t)]^2\}^{1/2}} - f'_1(x_3) \int_{x_3}^{a^*} \frac{f'_1(t) dt}{\{1 + [f'_1(t)]^2\}^{1/2}}, \end{aligned}$$

$$\sum_{\nu=1}^4 A_{1\nu} = -u_4 = - \int_0^{a^*} \frac{[f'_1(t)]^2 dt}{\{1 + [f'_1(t)]^2\}^{1/2}},$$

$$\begin{aligned} \sum_{\nu=1}^2 A_{2\nu} &= f_1(x_2) + (a^* - x_2)f'_1(x_2), \quad \sum_{\nu=1}^3 A_{2\nu} = f_1(x_3) + (a^* - x_3)f'_1(x_3), \\ \sum_{\nu=1}^4 A_{2\nu} &= f_1(a^*), \quad \sum_{\nu=1}^2 A_{3\nu} = f'_1(x_2), \quad \sum_{\nu=1}^3 A_{3\nu} = f'_1(x_3), \quad \sum_{\nu=1}^4 A_{3\nu} = f'_1(a^*). \end{aligned}$$

Consider the determinant

$$\begin{aligned} \delta_1 &= \det \left\{ \left(\sum_{\nu=1}^s A_{j\nu} \right) \begin{matrix} j = 2, 3 \\ s = 3, 4 \end{matrix} \right\} = [f_1(x_3) + (a^* - x_3)f'_1(x_3)]f'_1(a^*) - \\ &= f_1(a^*)f'_1(x_3) - f'_1(x_3)f'_1(a^*) \left\{ \frac{f_1(x_3)}{f'_1(x_3)} + a^* - x_3 - \frac{f_1(a^*)}{f'_1(a^*)} \right\} \end{aligned}$$

($0 < f'_1(x_3) < f'_1(a^*)$) by the hypothesis of the lemma and the choice of the points x_1 , x_2 and x_3). The second factor in the right part of the last

equalities subject to the following transformations:

$$\begin{aligned}
\frac{\delta_1}{f_1'(a^*)f_1'(x_3)} &= \frac{f_1(x_3)}{f_1'(x_3)} - \frac{f_1(a^*)}{f_1'(a^*)} + a^* - x_3 = \\
&- \frac{f_1(a^*) - f_1(x_3)}{f_1'(a^*)} - f_1(x_3) \left(\frac{1}{f_1'(a^*)} - \frac{1}{f_1'(x_3)} \right) + a^* - x_3 = \\
&- \frac{f_1'(\theta)(a^* - x_3)}{f_1'(a^*)} - f_1(x_3) \frac{f_1'(x_3) - f_1'(a^*)}{f_1'(a^*)f_1'(x_3)} + a^* - x_3 = \\
&- \left\{ \frac{f_1'(\theta) - f_1'(a^*)}{f_1'(x_3) - f_1'(a^*)} (a^* - x_3) + \frac{f_1(x_3)}{f_1'(x_3)} \right\} \frac{f_1'(x_3) - f_1'(a^*)}{f_1'(a^*)} \quad (5.5)
\end{aligned}$$

($x_3 < \theta < a^*$). The convexity of f_1 implies that

$$\left| \frac{f_1'(\theta) - f_1'(a^*)}{f_1'(x_3) - f_1'(a^*)} \right| \leq 1.$$

Therefore, if the condition

$$a^* - x_3 < \frac{f_1(x_3)}{f_1'(x_3)} \quad (5.6)$$

holds then $\delta \neq 0$ and, consequently, $\text{rank } \tilde{N} = \text{rank } N \geq 2$.

Analogously, we can establish that if the point x_1 is fixed and the point x_2 ($> x_1$) is so near to x_1 that the condition

$$x_2 - x_1 < \frac{f_1(x_1)}{f_1'(x_1)} \quad (5.7)$$

holds, then

$$\begin{aligned}
\delta_2 = \det \left\{ \left(\sum_{\nu=1}^s A_{j\nu} \right) \begin{matrix} j = 2, 3 \\ s = 1, 2 \end{matrix} \right\} &= \{f_1(x_1) + (a^* - x_1)f_1'(x_1)\}f_1'(x_2) - \\
f_1'(x_1)\{f_1(x_2) + (a^* - x_2)f_1'(x_2)\} &= f_1'(x_1)f_1'(x_2) \left\{ \frac{f_1(x_1)}{f_1'(x_1)} - \frac{f_1(x_2)}{f_1'(x_2)} + \right. \\
&\quad \left. x_2 - x_1 \right\} \neq 0.
\end{aligned}$$

If we suppose that the first row of the matrix \tilde{N} is a linear combination of two other rows of \tilde{N} then we obtain two pairs of the relations

$$-\frac{u_3}{f_1'(x_3)} - v_3 = C_1 \left\{ \frac{f_1(x_3)}{f_1'(x_3)} + (a^* - x_3) \right\} + C_2,$$

$$-\frac{u_4}{f_1'(a^*)} = C_1 \frac{f_1(a^*)}{f_1'(a^*)} + C_2$$

and

$$\begin{aligned} -\frac{u_1}{f_1'(x_1)} - v_1 &= C_1 \left\{ \frac{f_1(x_1)}{f_1'(x_1)} + (a^* - x_1) \right\} + C_2, \\ -\frac{u_2}{f_1'(x_2)} - v_2 &= C_1 \left\{ \frac{f_1(x_2)}{f_1'(x_2)} + (a^* - x_2) \right\} + C_2, \end{aligned}$$

from which it follows that, under the realization of the conditions (5.6) and (5.7),

$$\begin{aligned} C_1 &= \left\{ -\frac{u_3}{f_1'(x_3)} + \frac{u_4}{f_1'(a^*)} - v_3 \right\} / \left\{ \frac{f_1(x_3)}{f_1'(x_3)} - \frac{f_1(a^*)}{f_1'(a^*)} + (a^* - x_3) \right\} = \\ &= \left\{ -\frac{u_2}{f_1'(x_2)} + \frac{u_1}{f_1'(x_1)} - v_2 + v_1 \right\} / \left\{ \frac{f_1(x_2)}{f_1'(x_2)} - \frac{f_1(x_1)}{f_1'(x_1)} - (x_2 - x_1) \right\}, \quad (5.8) \end{aligned}$$

moreover, C_1 does not depend on the location of the points x_1, x_2 and x_3 .

Further, we have

$$\begin{aligned} \frac{u_4}{f_1'(a^*)} - \frac{u_3}{f_1'(x_3)} - v_3 &= \frac{u_4 - u_3}{f_1'(a^*)} + u_3 \left(\frac{1}{f_1'(a^*)} - \frac{1}{f_1'(x_3)} \right) - v_3 = \\ &= \left\{ -\frac{u_3}{f_1'(a^*)f_1'(x_3)} + O(a^* - x_3) \right\} \{f_1'(a^*) - f_1'(x_3)\}. \end{aligned}$$

Note that, here, we used the following estimates:

$$\begin{aligned} \left| \frac{u_4 - u_3}{f_1'(a^*)} - v_3 \right| &= \left| \frac{1}{f_1'(a^*)} \int_{x_3}^{a^*} \frac{[f_1(t)]^2 dt}{\{1 + [f_1'(t)]^2\}^{1/2}} - \int_{x_3}^{a^*} \frac{f_1'(t) dt}{\{1 + [f_1'(t)]^2\}^{1/2}} \right| = \\ &= \frac{1}{f_1'(a^*)} \left| \int_{x_3}^{a^*} \frac{f_1'(t)[f_1'(t) - f_1'(a^*)] dt}{\{1 + [f_1'(t)]^2\}^{1/2}} \right| \leq \frac{f_1'(a^*) - f_1'(x_3)}{f_1'(a^*)} \int_{x_3}^{a^*} \frac{f_1'(t) dt}{\{1 + [f_1'(t)]^2\}^{1/2}} \\ &\leq \frac{f_1'(a^*) - f_1'(x_3)}{f_1'(a^*)} \frac{f_1'(a^*)}{\{1 + [f_1'(a^*)]^2\}^{1/2}}. \end{aligned}$$

From these calculations and (5.5), we will obtain, as a result, the equality

$$\begin{aligned} C_1 &= \lim_{x_3 \rightarrow a^*} \left\{ -\frac{u_3}{f_1'(x_3)} + \frac{u_4}{f_1'(a^*)} - v_3 \right\} / \left\{ \frac{f_1(x_3)}{f_1'(x_3)} - \frac{f_1(a^*)}{f_1'(a^*)} + (a^* - x_3) \right\} = \\ &= -\frac{u_4}{f_1'(a^*)} \neq 0. \quad (5.9) \end{aligned}$$

By the analogy with (5.9) and on the basis of (5.8), we can also to establish that

$$C_1 = -\frac{u_1}{f_1(x_1)}.$$

And by the convexity downwards of the function f_1 ,

$$\begin{aligned} \frac{u_1}{f_1(x_1)} &= \frac{1}{f_1(x_1)} \int_0^{x_1} \frac{[f_1'(t)]^2 dt}{\{1 + [f_1'(t)]^2\}^{1/2}} = \\ &= \frac{1}{f_1(x_1)} \frac{f_1'(x_1)}{\{1 + [f_1'(x_1)]^2\}^{1/2}} \int_\xi^{x_1} f_1'(t) dt = \frac{f_1'(x_1)}{\{1 + [f_1'(x_1)]^2\}^{1/2}} \frac{f_1(x_1) - f_1(\xi)}{f_1(x_1)} \\ &= \frac{f_1'(x_1)}{\{1 + [f_1'(x_1)]^2\}^{1/2}} \frac{f_1(x_1) - f_1(\xi)}{f_1(x_1) - f_1(0)} \leq \frac{f_1'(x_1)}{\{1 + [f_1'(x_1)]^2\}^{1/2}} \rightarrow_{x_1 \rightarrow 0} 0 \end{aligned}$$

($0 < \xi < x_1$), therefore, $C_1 = 0$. As the result of that, we have the contradiction with (5.9). The latter, in turn, leads to the relations $\text{rank } N = \text{rank } \tilde{N} = 3$. The proof of the lemma is completed.

In conclusion, note that the main results of our article were earlier announced in [6], [16], [12], [18], [19].

Acknowledgements

The author was partially supported by the Russian Foundation for Basic Research (Grant 11-01-00819-a), the Interdisciplinary Project of the Siberian and Far-Eastern Divisions of the Russian Academy of Sciences (2012-2014 no. 56), the State Maintenance Program for the Leading Scientific Schools of the Russian Federation (Grant NSH-921.2012.1) and the Exchange Program between the Russian and Polish Academies of Sciences (Project 2014-2016).

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